HOMEWORK 6

Due on Monday, May 11th, in class.

Exercise 1. (10 points) Assume that the sets A and B are separated and let $A_1 \subseteq A$ and $B_1 \subseteq B$. Prove that A_1 and B_1 are separated.

Exercise 2. (10 points) Assume that the sets A and B are separated and that the sets A and C are separated. Prove that the sets A and $B \cup C$ are separated.

Exercise 3. (10 points) If A and B are closed sets, prove that $A \setminus B$ and $B \setminus A$ are separated.

Exercise 4. (10 points) Let (X, d) be a connected metric space and let A be a connected subset of X. Assume that the complement of A is the union of two separated sets B and C. Prove that $A \cup B$ and $A \cup C$ are connected. Prove also that if A is closed, then so are $A \cup B$ and $A \cup C$.

Exercise 5. (10 points) Let (X, d) be a metric space and let A, B be two closed subsets of X such that $A \cup B$ and $A \cap B$ are connected. Prove that A is connected.

Exercise 6. (10 points) Let $\{A_i\}_{i \in I}$ be a family of connected sets such that one set of the family intersects all the others. Prove that $\cup_{i \in I} A_i$ is connected.

Exercise 7. (20 points) Let (X, d) be a metric space and let $A \subseteq X$. Prove that the following statements are equivalent:

- A is disconnected.
- There exist D_1, D_2 open subsets in X such that $A \subseteq D_1 \cup D_2$, $A \cap D_1 \neq \emptyset$, $A \cap D_2 \neq \emptyset$, but $A \cap D_1 \cap D_2 = \emptyset$.
- There exist F_1, F_2 closed subsets in X such that $A \subseteq F_1 \cup F_2$, $A \cap F_1 \neq \emptyset$, $A \cap F_2 \neq \emptyset$, but $A \cap F_1 \cap F_2 = \emptyset$.

Exercise 8. (10 points) Let (X, d) be a metric space and let $A \subseteq X$. Prove that if every pair of points in A belong to a connected subset of A, then A is connected.

Exercise 9. (10 points) Let (X, d) be a metric space and let A, B be two subsets of X such that $A \cap B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Assume also that B is connected. Prove that $Fr(A) \cap B \neq \emptyset$. Deduce that if X is connected, then every subset of X, other than $\emptyset$ or X, has at least one frontier point.