HOMEWORK 3

Due on Monday, April 20th, in class.

Exercise 1. (10 points) Let (X, d) be a metric space and let A, B be two subsets of X. Prove that
$$A° \cup B° \subseteq (A \cup B)° \quad \text{and} \quad A° \cap B° = (A \cap B)°.$$

Exercise 2. (10 points) Let (X, d) be a metric space and let A, B be two subsets of X. Prove that
$$\bar{A} \cup \bar{B} = \overline{A \cup B} \quad \text{and} \quad \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}.$$

Exercise 3. (10 points) Let (X, d) be a metric space and let A be a subset of X. Prove that a point $x \in X$ is an adherent point of A if and only if $d(x, A) = 0$.

Exercise 4. (10 points) Let (X, d) be a metric space and let A be a subset of X. Prove that the diameter of A is equal to the diameter of the closure of A, that is, $\delta(A) = \delta(\bar{A})$.

Exercise 5. (10 points) Let (X, d) be a metric space and let A be a subset of X and O be an open subset of X. Prove that
$$O \cap \bar{A} \subseteq \overline{O \cap A} \quad \text{and} \quad \overline{O \cap \bar{A}} = \overline{O \cap A}.$$
Conclude that if $O \cap A = \emptyset$, then $O \cap \bar{A} = \emptyset$.

Exercise 6. (10 points) Let (X, d) be a metric space and let $a \in X$ and $r > 0$. Prove that the closed ball $B_r(a) = \{x \in X \,|\, d(x, a) \leq r\}$ is a closed set.

Exercise 7. (10 points) Let (X, d) be a metric space and let A, B be two subsets of X. Prove that
$$Fr(A \cup B) \subseteq Fr(A) \cup Fr(B).$$
Show also that if $\bar{A} \cap \bar{B} = \emptyset$, then $Fr(A \cup B) = Fr(A) \cup Fr(B)$.

Exercise 8. (10 points) Let (X, d) be a metric space and let A be a subset of X. Prove that
$$Fr(\bar{A}) \subseteq Fr(A)$$
$$Fr(A°) \subseteq Fr(A)$$
$$\bar{A} = A° \cup Fr(A).$$

Exercise 9. (10 points) Let (X, d) be a metric space and let A be a subset of X. Prove that A is closed if and only if $Fr(A) \subseteq A$.

Exercise 10. (10 points) Let (X, d) be a metric space and let A be a subset of X. Prove that A is open if and only if $Fr(A) \cap A = \emptyset$.