Homework 1 for Math 216 Geometric Invariant Theory, UCLA

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Due: Wednesday, February 20, 2013

(1) Show that the geometric quotient $A^2/\{\pm 1\}$ exists and describe it explicitly. Show that it has exactly one singular point, the image of the origin. (*) Show how to resolve this surface singularity.

(2) Show that the affine categorical quotient of the space of $n \times n$ complex matrices by the conjugation action of $GL_n\mathbf{C}$ is isomorphic to affine space $\mathbf{C}^n$, with the map $M_n(\mathbf{C}) \to \mathbf{C}^n$ given by the characteristic polynomial of a matrix.

(3) Consider the action of $SL_2\mathbf{C}$ on the space $(\mathbf{P}^1)^n$ of ordered n-tuples of points in $\mathbf{P}^1$. Show that a point $(x_1, \ldots, x_n)$ is semistable if and only if at most $n/2$ of the points are equal. Show that it is stable if and only less than $n/2$ of the points are equal (to any given point in $\mathbf{P}^1$). Show that $PGL_2\mathbf{C} = SL_2\mathbf{C}/\{\pm 1\}$ acts freely on the open subset of stable points in $(\mathbf{P}^1)^n$.

(4) Using (3), show that for n odd, the geometric quotient X of the stable points by $SL_2\mathbf{C}$ exists and is a smooth projective variety. (*) For $n = 5$, describe this variety explicitly. (It has complex dimension 2.) For example, try to compute its cohomology, $H^*(X, \mathbf{Q})$; this will require a fairly explicit geometric description.

(5) Show that any homomorphism of complex algebraic groups from the multiplicative group G_m to itself has the form $z \mapsto z^n$ for some integer n.

(6) Give an example to show that a sub-$\mathbf{C}$-algebra of a polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$ need not be finitely generated as a $\mathbf{C}$-algebra. Justify your answer.

(7) Let G be a complex reductive group acting on a projective variety X with a G-equivariant ample line bundle L. (Hence, using a suitable positive power of L, we have a G-equivariant embedding $X \hookrightarrow P(V)$ for some representation V of G.) These data determine open sets of stable and semistable points in X, and a GIT quotient $X//G$.

Show that if x is a semistable point in X, then x is stable if and only if the orbit of x is closed in X^{ss} and the orbit of x in X^{ss} has dimension equal to $\dim(G)$. (Thus, given the G-action on X (but not the equivariant line bundle), the semistable set determines the stable set.) Beware that the orbit of a stable point need not be closed in X (give an example).