# Homework 8 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1) Let $k$ be a field. Compute $\mathrm{Ext}^i_{k[x]/(x^2)}(k,k)$ for all $i$. Deduce that $k$ does not have a finite projective resolution as a $k[x]/(x^2)$-module. (First, say why there is only one way to view $k$ as a $k[x]/(x^2)$-module, compatibly with the obvious structure of $k$ as a $k$-module.)

(2) Let $R$ be an algebra of finite type over a field $k$. Assume that $R$ is a domain. Let $M$ be a finitely generated module over $R$. Show that $M$ is projective (or equivalently, flat or locally free) if and only if the $R/\mathfrak{m}$-vector spaces $M/\mathfrak{m}M$ all have the same dimension. (Hint: If these vector spaces have dimension $n$, construct a surjection $R_{\mathfrak{m}}^{\oplus n} \to M_{\mathfrak{m}}$ at any maximal ideal $\mathfrak{m}$. Show that this comes from a surjection $R[1/s]^{\oplus n} \to M[1/s]$ for some $s \in R - \mathfrak{m}$. Show that any element of the kernel must be zero.)

As an application, show that the ring $k[x,y]/(x^2, xy)$ is finite but not flat over $k[y]$.

(3) Show that every finitely generated module $M$ over a Dedekind domain $R$ is the direct sum of a torsion module $M_1$ and a projective module $M_2$. Give an example to show that such a splitting is not unique (in the sense that the submodules $M_1$ and $M_2$ of $M$ are not uniquely determined).

(4) Let $A$ and $B$ be algebras of finite type over a field $k$. Show that $\dim(A \otimes_k B) = \dim(A) + \dim(B)$.