Homework 7 for Math 215A Commutative Algebra

Burt Totaro

Due: Monday, November 12, 2012

Rings are understood to be commutative, unless stated otherwise.

(1) Let $f : X \to Y$ be a morphism of affine varieties over a field k. By definition, f is *dominant* if it has dense image. Show that f is dominant if and only if the pullback map $f^* : O(Y) \to O(X)$ on regular functions is injective. (Remember that an affine variety has a generic point.)

(2) Let k be a field. Define $f : A^2_k \to A^2_k$ by $f(x, y) = (x, xy)$. (Explain what this means in terms of rings.) Show that the morphism f of affine varieties over k is birational but that there is a non-closed point of A^2_k which f maps to a closed point. Also, give an example of a morphism of affine schemes which maps a closed point to a non-closed point.

(3) Let R be an algebra of finite type over a field k. Show that R is artinian if and only if R has finite dimension as a k-vector space. (Hint: consider the filtration of R constructed in the proof that "artinian" is equivalent to "noetherian of dimension zero".)

(4) (a) Let $\mathfrak{p}$ be a prime ideal in a noetherian ring R, and let M and N be R-modules. Give an example to show that the natural map

$$\mathrm{Hom}_R(M, N)_{\mathfrak{p}} \to \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

need not be an isomorphism. But show that it is an isomorphism if M is finitely generated.

(b) Let $\mathfrak{p}$ be a prime ideal in a noetherian ring R. Let M and N be R-modules with M finitely generated. Show that $\mathrm{Ext}^i_R(M, N)_{\mathfrak{p}} \cong \mathrm{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ for any $i \geq 0$.

(c) Show that a finitely generated module over a noetherian ring R which is locally free must be projective. Deduce that "flat = locally free = projective" for finitely generated modules over a noetherian ring. (In geometric language: finitely generated projective modules over a noetherian ring R are equivalent to vector bundles over the affine scheme $\mathrm{Spec}\,(R)$.)