Homework 6 for Math 215A Commutative Algebra

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Due: Monday, November 5, 2012

Rings are understood to be commutative, unless stated otherwise.

(1) Find the irreducible components of the closed subset $X = \{(x, y, z) \in A^3_{\mathbf{C}} : x^2 = yz, xz = x\}$. Justify your answer. (Recall that affine space A^n_k means $\operatorname{Spec} k[x_1, \ldots, x_n]$.)

(2) Let k be a field, and let R be the polynomial ring $k[x, y]$. Give a basis for the R-module $M = R/(x^2, xy)$ as a k-vector space. Find a filtration of the R-module M into finitely many subquotients of the form $R/\mathfrak{p}_i$ with $\mathfrak{p}_i$ prime ideals. What is the support of M, as a closed subset of A^2_k?

(3) Show that every factorial domain (also called a unique factorization domain) is normal. So, for example, the integers $\mathbf{Z}$ and polynomial rings over a field are normal.

(4) Compute the ring of integers of the number field $\mathbf{Q}(\sqrt{5})$. (By definition, this means the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{5})$.)

(5) Let k be a field. Show that the k-algebra $k[x, y]/(x^2 = y^3)$ is a domain but is not normal. What is its normalization? Also, give an example of a domain which is a finite flat $\mathbf{Z}$-algebra but is not normal.