Homework 5 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1) (Snake lemma) Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0 \end{array}$$

is a commutative diagram of R-modules with exact rows. The snake lemma says that there is a long exact sequence of R-modules:

$$0 \to \ker f \to \ker g \to \ker h \to \text{coker } f \to \text{coker } g \to \text{coker } h \to 0.$$

Do the following steps of the proof. (a) Define the "boundary map" $\ker h \to \text{coker } f$. (The other maps in the sequence should be clear.) Show that your definition of the boundary map is independent of choices in your construction. (b) Show that the sequence is exact at $\ker h$. (You should be able to check exactness of the sequence at any step, upon request.)

(2) Show that the following are equivalent, for a module M over a ring R. (1) M is projective. (2) $\text{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i > 0$. (3) $\text{Ext}^1_R(M, N) = 0$ for all R-modules N. Likewise, show that the following are equivalent, for a module M over a ring R. (1) M is flat. (2) $\text{Tor}^R_i(M, N) = 0$ for all R-modules N and all $i > 0$. (3) $\text{Tor}^R_1(M, N) = 0$ for all R-modules N.

(3) (a) Show that homology commutes with direct limits. That is, suppose we are given a directed system of two-step complexes $B_\alpha \to C_\alpha \to D_\alpha$ of R-modules, for α running over a directed set A. (A "directed system of complexes" means a functor from a directed set A to the category of complexes.) Show that the homology of the complex $\varinjlim B_\alpha \to \varinjlim C_\alpha \to \varinjlim D_\alpha$ is isomorphic to the direct limit of the homology of the original complexes. (Use the universal property of the direct limit to define a map.)

(b) It is a general result of category theory that any functor which is a left adjoint (that is, which has a right adjoint) preserves all colimits which exist in the domain category. Deduce that tensor products commute with direct limits in each variable. That is, given a directed system of R-modules N_α, show that

$M \otimes_R \varinjlim N_\alpha$ is isomorphic to $\varinjlim(M \otimes_R N_\alpha)$. (The same argument shows that $(\varinjlim M_\alpha) \otimes_R N \cong \varinjlim(M_\alpha \otimes_R N)$.)

(c) Deduce that Tor commutes with direct limits in each variable.

(4) (a) Let M be a finitely generated flat module over a noetherian local ring R. Show that M is free. (Hint: Choose elements $x_1, \ldots, x_n$ of M which map to a basis for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M = M \otimes_R R/\mathfrak{m}$. By Nakayama's lemma, we know that $x_1, \ldots, x_n$ generate M as an R-module. Show that the resulting map $R^{\oplus n} \to M$ is an isomorphism.)

(b) Let M be a finitely generated module over a noetherian ring R. Show that the following are equivalent:

(1) M is flat;

(2) $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all prime ideals $\mathfrak{p}$ in R;

(3) $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ in R.

In short, "flat = locally free", for finitely generated modules over a noetherian ring.