Homework 4 for Math 215A Commutative Algebra

Burt Totaro

Due: Monday, October 22, 2012

Rings are understood to be commutative, unless stated otherwise.

(1) A *finite* algebra over a ring A means an A-algebra which is finitely generated as an A-module. Show that a finite flat $\mathbf{Z}$-algebra is free as a $\mathbf{Z}$-module, and give an example of a flat $\mathbf{Z}$-algebra which is not free as a $\mathbf{Z}$-module. Is a flat $\mathbf{Z}$-algebra of finite type always free as a $\mathbf{Z}$-module?

(2) Let k be a field. Show that every ideal in the polynomial ring $k[x]$ is generated by one element. But show that the ideal $(x^n, x^{n-1}y, \ldots, y^n)$ in $k[x,y]$ cannot be generated by fewer than $n+1$ elements. Thus there is no upper bound for the number of elements needed to generate an ideal in $k[x,y]$.

(3) Let R be a noetherian ring. We showed that $X = \operatorname{Spec}(R)$ can be written as the union of finitely many irreducible closed subsets, $X = X_1 \cup X_2 \cup \cdots \cup X_m$, such that X_i is not contained in X_j for any $i \neq j$. Show that such a decomposition of X is unique up to reordering the X_i's.

(4) Let M be a nonzero module over a noetherian ring R. Show that there is an ideal $\mathfrak{p}$ in R which is maximal among the annihilators of nonzero elements of M. Show that such an ideal $\mathfrak{p}$ must be prime.