Homework 3 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1) Let R be a factorial domain (that is, a UFD). Show that a principal ideal (f) in R is prime if and only if $f = 0$ or f is irreducible. (Thus we have a large class of examples of prime ideals in a factorial domain such as a polynomial ring $k[x_1, \ldots, x_n]$: the ideal $(f) \subset k[x_1, \ldots, x_n]$ is prime for any irreducible polynomial f over k.)

For ideals with more than one generator in a polynomial ring, primeness of the ideal is harder to read off from the generators. For example, find two irreducible polynomials f and g in $\mathbf{C}[x, y]$ such that the ideal (f, g) is not prime.

(2) Show that the kernel of the $\mathbf{C}$-algebra homomorphism $\mathbf{C}[x, y] \to \mathbf{C}[t]$ given by $x \mapsto t^2$ and $y \mapsto t^3$ is the ideal $(x^3 - y^2)$. (One possible approach is to show first that every element of the quotient ring $\mathbf{C}[x, y]/(x^3 - y^2)$ can be written as $f(x) + g(x)y$ for some polynomials f and g.) Deduce that the ideal $(x^3 - y^2)$ in $\mathbf{C}[x, y]$ is prime.

(3) For any commutative ring R, show that $\operatorname{Spec}(R)$ is quasi-compact. (That is, if $\operatorname{Spec}(R)$ is the union of some collection of open subsets, then it is the union of finitely many of them. In point-set topology this would just be called "compact". The word "quasi-compact" is meant to emphasize that these topological spaces are not necessarily Hausdorff.)

(4) Let R be a nonzero commutative ring. Let I and J be sets of different cardinalities. Show that the free R-modules $R^{\oplus I}$ and $R^{\oplus J}$ are not isomorphic. (Hint: this is true when R is a field.)

(5) Let I be an ideal in a commutative ring R, $I \neq R$. Show that there is a minimal prime ideal containing I. (That means: there is a prime ideal containing I which contains no other prime ideal containing I.) What does this mean geometrically, in terms of $\operatorname{Spec}(R)$?