Homework 10 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1) Let f be an irreducible polynomial over an algebraically closed field k. Let P be a closed point of the variety $X = \{f = 0\}$ in affine space A^n over k, and let $\mathfrak{m}$ be the corresponding maximal ideal in $R = k[x_1, \ldots, x_n]/(f)$. Show that X is smooth at P if and only if $R_{\mathfrak{m}}$ is a regular local ring. (Start by showing that R has dimension $n - 1$.)

(2) Find all the closed points at which the 1-dimensional affine scheme $\{x^2 + y^2 + z^2 = 1,\ xz - 2y + 2 = 0\}$ in $A^3_{\mathbf{C}}$ is not smooth over $\mathbf{C}$.

(3) Let $f : X \to Y$ be a finite morphism of affine schemes. (That means that $X = \text{Spec}\,(R)$ and $Y = \text{Spec}\,(S)$ for some commutative rings R and S, f is the mapping associated to a ring homomorphism $S \to R$, and R is a finite S-algebra.) Show that the inverse image of each point in Y is a finite subset of X (that is, a finite morphism is *quasi-finite*). (Try to reduce the problem as much as possible, to the case where S is a domain or even a field.)

Show that the image of any closed subset of X is closed in Y (that is, a finite morphism is *closed*). Give an example of a morphism of affine schemes which is (a) quasi-finite but not closed; (b) closed but not quasi-finite.

(4) (a) Let R be a ring. If $R_{\mathfrak{m}}$ is noetherian for every maximal ideal $\mathfrak{m}$ in R and every nonzero element of R is contained in only finitely many maximal ideals, show that R is noetherian.

(b) Let $A = \mathbf{C}[x_1, x_2, \ldots]$ be the polynomial ring on a countably infinite set of generators. Let $m_1, m_2, \ldots$ be a strictly increasing sequence of positive integers such that the differences $m_2 - m_1, m_3 - m_2, \ldots$ are also strictly increasing. Let $\mathfrak{p}_i = (x_{m_i+1}, \ldots, x_{m_{i+1}})$ and let $S = A - \cup_{i \geq 1} \mathfrak{p}_i$. Show that $R = A[S^{-1}]$ is a noetherian ring which does not have finite dimension. (Use (a).)