Homework 3 for Math 214B Algebraic Geometry

Burt Totaro

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Due on Monday, May 14.

(1) For a hypersurface X of degree d in $\mathbf{P}^{n+1}$ over a field k, $n \geq 1$, compute the Hilbert series of the graded ring $\oplus_j H^0(X, O(j))$. Write the series as a rational function.

(2) Let X be a hypersurface of degree d in projective space $\mathbf{P}^{n+1}$ over a field k. Compute the cohomology groups $H^i(X, O_X)$ for all i. Deduce that X is not isomorphic to $\mathbf{P}^n$ for d large enough; what range of d do you get?

(3) Let X be the affine line over a field k. We know that $H^i(X, E) = 0$ for every quasi-coherent sheaf E on X (or any affine scheme) and every $i > 0$. Does this vanishing hold for every sheaf of O_X-modules, not necessarily quasi-coherent?

(4) Let $U = A^2 - 0$ over a field k. Using a suitable cover of U by affine open subsets, show that $H^1(U, O)$ is isomorphic to the k-vector space with basis $\{x^i y^j : i, j < 0\}$. In particular, it is a k-vector space of infinite dimension. Use this calculation to show that the scheme U is not affine.

(5) Let X be a noetherian separated scheme. Define the *cohomological dimension* of X, denoted $\mathrm{cd}(X)$, to be the least integer n such that $H^i(X, F) = 0$ for all quasi-coherent sheaves F and all $i > n$. For example, Serre's Theorem III.3.7 in Hartshorne says that $\mathrm{cd}(X) = 0$ if and only if X is affine. Grothendieck's Theorem III.2.7 implies that $\mathrm{cd}(X) \leq \dim(X)$.

(a) In the definition of $\mathrm{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use exercise II.5.15 and Prop. III.2.9.

(b) If X is quasi-projective over a field k, then it is even sufficient to consider vector bundles on X. Use Cor. II.5.18.

(c) Suppose that X has a covering by $r + 1$ open affine subsets. Use Cech cohomology to show that $\mathrm{cd}(X) \leq r$.

(d) If X is quasi-projective scheme of dimension r over a field k, show that X can be covered by $r + 1$ open affine subsets. Conclude (independent of Grothendieck's theorem) that $\mathrm{cd}(X) \leq \dim(X)$.

(e) Let Y be a set-theoretic complete intersection (exercise I.2.17) of codimension r in $X = A^n_k$. Show that $\mathrm{cd}(X - Y) \leq r - 1$.

(6) Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine 4-space over a field k. Let Y_1 be the

plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y} \subset \mathbf{P}^4_k$ is also not a set-theoretic complete intersection. [Hint: Use problem 5(e) above. Then show that $H^2(X - Y, O_X) \neq 0$, by using exercises III.2.3 (cohomology with support) and III.2.4 (Mayer-Vietoris).]