Homework 2 for Math 214B Algebraic Geometry

Burt Totaro

Spring 2018, UCLA

Due on Monday, April 30.

(1) If $X \subset \mathbf{P}^n$ over a field k is any hypersurface, show that $\mathbf{P}^n - X$ is affine. [Hint: Let X have degree d. Then consider the dth Veronese embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$. What does $\mathbf{P}^n - X$ look like in terms of that embedding?]

(2) Show that any two curves in $\mathbf{P}^2$ over a field k have nonempty intersection. More generally, show that if $X \subset \mathbf{P}^n$ is a projective variety of dimension ≥ 1, and if $D \subset \mathbf{P}^n$ is a hypersurface, then $X \cap D \neq \emptyset$. [Hint: use that $\mathbf{P}^n - D$ is affine, by the previous problem. Depending on your approach, you may want to prove the result first when k is algebraically closed and then deduce the general statement.]

(3) (a) Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial. Suppose that the subset of $\{f = 0\} \subset \mathbf{P}^n$ defined by $\partial f/\partial x_0 = 0, \ldots, \partial f/\partial x_n = 0$ is empty (so $\{f = 0\}$ is a smooth hypersurface in $\mathbf{P}^n$). If $n \geq 2$, show that the polynomial f is irreducible over k and hence the hypersurface $\{f = 0\} \subset \mathbf{P}^n$ is irreducible. [Hint: use the previous problem.]

(b) Let k be an algebraically closed field of characteristic zero. For any $d \geq 1$, show that the Fermat curve $x^d + y^d + z^d = 0$ in $\mathbf{P}^2_k$ is smooth and irreducible.

(4) Let X be a projective variety over an algebraically closed field k, L a line bundle on X. If L and L^* are both effective (that is, their spaces of sections are nonzero), show that L is trivial.

(5) A *divisor with simple normal crossings* (snc) on a regular scheme X means a union of regular irreducible divisors $D_1, \ldots, D_r$ in X which all intersect transversely. (Equivalently, for any s, the scheme-theoretic intersection of any s of the divisors is regular of codimension s in X, or else empty.)

Let C be the cuspidal cubic curve $x^2 = y^3$ in the smooth surface $X = A^2_{\mathbf{C}}$. By repeatedly blowing up points, construct a smooth surface W over $\mathbf{C}$ with a proper birational morphism $f\colon W \to X$ such that the closed subset $f^{-1}(C)$ is an snc divisor. Draw a picture of the inverse image of C at each stage of your blow-up (making sure to distinguish transverse from non-transverse intersections).

(6) A *resolution of singularities* of an integral scheme X means a regular integral scheme W with a proper birational morphism $W \to X$. If X is the affine cone over a smooth projective variety Y over a field k, then W can be produced by one blow-up

at a point; the exceptional divisor is then $\cong Y$. For example, resolve the singularities of the affine quadric cone $xy = z^2$ in $A^3_{\mathbf{C}}$, and draw the exceptional divisor.

For a more interesting example, resolve the singularities of the surface singularity $xy = z^3$ over $\mathbf{C}$, by blowing up points as needed. Again, draw the exceptional divisor of your resolution.

(7) Let k be a field. Classify all varieties over k which are both affine and projective over k.

(8) Let X be a smooth complete intersection in $\mathbf{P}^n$ over a field k. (That is, X is a smooth subscheme of some dimension r in $\mathbf{P}^n$, and X is an intersection $X = Z_1 \cap \cdots \cap Z_{n-r}$ of hypersurfaces as a scheme. If necessary, assume that $Z_1, \ldots, Z_{n-r}$ are smooth. Note that in any case we know that $Z_1, \ldots, Z_{n-r}$ are smooth and transverse *near* X, and ideally you could try to use only that.)

Compute the canonical bundle of any smooth complete intersection curve in any $\mathbf{P}^n$, using the adjunction formula. You can assume that the hypersurfaces all have degree at least 2, since a hypersurface of degree 1 is just a hyperplane $\mathbf{P}^{n-1} \subset \mathbf{P}^n$. You can use your formula in various ways. For example, show that there are only finitely many multidegrees of smooth complete intersections of a given dimension r that are *Fano* (that is, $-K_X$ ample), and list them all for X of dimension at most 3.