Homework 1 for Math 214B Algebraic Geometry

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A *variety* over a field k means an integral separated scheme of finite type over k.

Due Monday, Apr. 16.

(1) Let k be a field. Decompose the scheme $X = \{(x, y, z) \in A^3_k : x^2 = yz, xz = x\}$ into its irreducible components.

(2) Let k be a field. Let $f : A^1_k \to A^1_k$ be an isomorphism of k-varieties. Prove that f is given by a polynomial of degree 1.

(3) Let k be a field. Show that the varieties A^1 and $A^1 - \{0\}$ are not isomorphic over k. Likewise for A^2 and $A^2 - \{0\}$, which is a bit different.

(4) Let $X = \{(x, y) \in A^2 : x^2 = y^3\}$ over a field k. Define a bijective morphism from A^1 to X over k. Show that this is not an isomorphism. In fact, show that A^1 and X are not isomorphic over k.

(5) Let k be an algebraically closed field of characteristic zero. Find the singular points of the affine curve $xy + x^3 + y^3 = 0$ over k.

(6) Show that the conic $xy = z^2$ in $\mathbf{P}^2$ over any field k is isomorphic to $\mathbf{P}^1$. Show that every conic (irreducible curve of degree 2 in $\mathbf{P}^2$) over an algebraically closed field of characteristic not 2 can be moved to $xy = z^2$ by some automorphism of $\mathbf{P}^2$.

(7) *Linear subspaces of* $\mathbf{P}^n$. Let k be a field. A hypersurface in $\mathbf{P}^n$ over k of degree 1 (that is, defined by a homogeneous polynomial of degree 1) is called a *hyperplane*. A nonempty intersection of hyperplanes is called a *linear subspace* of $\mathbf{P}^n$.

(a) If Y is a linear subspace of dimension r in $\mathbf{P}^n$, show that Y is isomorphic to $\mathbf{P}^r$.

(b) Let Y, Z be linear subspaces of dimension r and s in $\mathbf{P}^n$. If $r + s - n \geq 0$, then $Y \cap Z \neq \emptyset$. Moreover, $Y \cap Z$ is a linear subspace of dimension at least $r + s - n$ in $\mathbf{P}^n$. (Think of $A^{n+1}(k)$ as a vector space over k, and work with its subspaces.)

(8) *The Veronese embedding.* Let k be a field. For a given $n, d > 0$, let $M_0, \ldots, M_N$ be all the monomials of degree d in the $n + 1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{d} - 1$. We define a morphism $\rho_d : \mathbf{P}^n \to \mathbf{P}^N$ over k by sending a

point $P = [a_0, \ldots, a_n]$ to the point $[M_0(a), \ldots, M_N(a)]$ obtained by substituting the numbers a_i in the monomials M_j. This is called the dth *Veronese embedding*, or the *d-uple embedding*, of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, when $n = 1$ and $d = 2$, this is the embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ as a conic.

(a) Show that the dth Veronese map of $\mathbf{P}^1$, $\rho_d : [u, v] \mapsto [u^d, u^{d-1}v, \ldots, v^d]$, is a morphism from $\mathbf{P}^1$ to $\mathbf{P}^d$. Show that the image is closed in $\mathbf{P}^d$. Show that ρ_d is an isomorphism from $\mathbf{P}^1$ to this closed subset, called a *rational normal curve* in $\mathbf{P}^d$. (These results hold for the Veronese embeddings of $\mathbf{P}^n$ for any n, but it takes longer to write out the proofs in general.)

(b) Show that the rational normal curve in $\mathbf{P}^3$ (called the *twisted cubic curve*) is the projective closure of the affine curve $\{(t, t^2, t^3)\} \subset A^3$, in some coordinates.