# Homework 1 for Math 214B Algebraic Geometry

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Spring 2017, UCLA

Due on Monday, April 24.

(1) Let $X$ be a projective variety over an algebraically closed field $k$, $L$ a line bundle on $X$. If $L$ and $L^*$ are both effective (that is, their spaces of sections are nonzero), show that $L$ is trivial.

(2) A *divisor with simple normal crossings* (snc) on a regular scheme $X$ means a union of regular irreducible divisors $D_1, \ldots, D_r$ in $X$ which all intersect transversely. (Equivalently, for any $s$, the scheme-theoretic intersection of any $s$ of the divisors is regular of codimension $s$ in $X$, or else empty.)

Let $C$ be the cuspidal cubic curve $x^2 = y^3$ in the smooth surface $X = A^2_{\mathbf{C}}$. By repeatedly blowing up points, construct a smooth surface $W$ over $\mathbf{C}$ with a proper birational morphism $f\colon W \to X$ such that the closed subset $f^{-1}(C)$ is an snc divisor. Draw a picture of the inverse image of $C$ at each stage of your blow-up (making sure to distinguish transverse from non-transverse intersections).

(3) A *resolution of singularities* of an integral scheme $X$ means a regular integral scheme $W$ with a proper birational morphism $W \to X$. If $X$ is the affine cone over a smooth projective variety $Y$ over a field $k$, then $W$ can be produced by one blow-up at a point; the exceptional divisor is then $\cong Y$. For example, resolve the singularities of the affine quadric cone $xy = z^2$ in $A^3_{\mathbf{C}}$, and draw the exceptional divisor.

For a more interesting example, resolve the singularities of the surface singularity $xy = z^3$ over $\mathbf{C}$, by blowing up points as needed. Again, draw the exceptional divisor of your resolution.

(3) Let $k$ be a field. Classify all varieties over $k$ which are both affine and projective over $k$.

(4) Let $X$ be a smooth complete intersection in $\mathbf{P}^n$ over a field $k$. (That is, $X$ is a smooth subscheme of some dimension $r$ in $\mathbf{P}^n$, and $X$ is an intersection $X = Z_1 \cap \cdots \cap Z_{n-r}$ of hypersurfaces as a scheme. If necessary, assume that $Z_1, \ldots, Z_{n-r}$ are smooth. Note that in any case we know that $Z_1, \ldots, Z_{n-r}$ are smooth and transverse *near* $X$, and ideally you could try to use only that.)

Compute the canonical bundle of any smooth complete intersection curve in any $\mathbf{P}^n$, using the adjunction formula. You can assume that the hypersurfaces all have degree at least 2, since a hypersurface of degree 1 is just a hyperplane $\mathbf{P}^{n-1} \subset \mathbf{P}^n$. You can use your formula in various ways. For example, show that there are only finitely many multidegrees of smooth complete intersections of a given dimension $r$ that are *Fano* (that is, $-K_X$ ample), and list them all for $X$ of dimension at most 3.

(5) Let $k$ be an algebraically closed field of characteristic $p > 0$. Show that the morphism $f : A^1_k \to A^1_k$ defined by $x \mapsto x^p$ is a bijective morphism but not an isomorphism. Where is the derivative of $f$ zero? Is $f : A^1 \to A^1$ birational?

(6) Define the *support* of a coherent sheaf $F$ on a scheme $X$ to be the intersection of all closed subsets $S \subset X$ such that $F|_{X-S} = 0$. Show that $\mathrm{supp}(F)$ is itself a closed subset such that $F|_{X-\mathrm{supp}(F)} = 0$. Show that for $X$ affine, $\mathrm{supp}(F)$ is the closed subset of $X$ corresponding to an ideal in $O(X)$, namely the *annihilator* of the $O(X)$-module of sections $F(X)$:

$$\operatorname{ann} H^0(X, F) = \{f \in O(X) : fs = 0 \in H^0(X, F) \text{ for all global sections } s \in F(X)\}.$$

This is sometimes a useful way to check that two sheaves $F_1, F_2$ on $X$ are not isomorphic: if they have different supports in $X$, they are not isomorphic.