Homework 1 for Math 214B Algebraic Geometry

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Due on Friday, April 22.

(1) Let X be a projective variety over an algebraically closed field k, L a line bundle on X. If L and L^* are both effective (that is, their spaces of sections are nonzero), show that L is trivial.

(2) A *divisor with simple normal crossings* (snc) on a regular scheme X means a union of regular irreducible divisors $D_1, \ldots, D_r$ in X which all intersect transversely. (Equivalently, for any s, the scheme-theoretic intersection of any s of the divisors is regular of codimension s in X, or else empty.)

Let C be the cuspidal cubic curve $x^2 = y^3$ in the smooth surface $X = A^2_{\mathbf{C}}$. By repeatedly blowing up points, construct a smooth surface W over $\mathbf{C}$ with a proper birational morphism $f\colon W \to X$ such that the closed subset $f^{-1}(C)$ is an snc divisor. Draw a picture of the inverse image of C at each stage of your blow-up (making sure to distinguish transverse from non-transverse intersections).

(3) A *resolution of singularities* of an integral scheme X means a regular integral scheme W with a proper birational morphism $W \to X$. If X is the affine cone over a smooth projective variety Y over a field k, then W can be produced by one blow-up at a point; the exceptional divisor is then $\cong Y$. For example, resolve the singularities of the affine quadric cone $xy = z^2$ in $A^3_{\mathbf{C}}$, and draw the exceptional divisor.

For a more interesting example, resolve the singularities of the surface singularity $xy = z^3$ over $\mathbf{C}$, by blowing up points as needed. Again, draw the exceptional divisor of your resolution.

(3) Let k be a field. Classify all varieties over k which are both affine and projective over k.

(4) Let X be a smooth complete intersection in $\mathbf{P}^n$ over a field k. (That is, X is a smooth subscheme of some dimension r in $\mathbf{P}^n$, and X is an intersection $X = Z_1 \cap \cdots \cap Z_{n-r}$ of hypersurfaces as a scheme. If necessary, assume that $Z_1, \ldots, Z_{n-r}$ are smooth. Note that in any case we know that $Z_1, \ldots, Z_{n-r}$ are smooth and transverse *near* X, and ideally you could try to use only that.)

Compute the canonical bundle of any smooth complete intersection curve in any $\mathbf{P}^n$, using the adjunction formula. You can assume that the hypersurfaces all have degree at least 2, since a hypersurface of degree 1 is just a hyperplane $\mathbf{P}^{n-1} \subset \mathbf{P}^n$. You can use your formula in various ways. For example, show that there are only finitely many multidegrees of smooth complete intersections of a given dimension r that are *Fano* (that is, $-K_X$ ample), and list them all for X of dimension at most 3.

(5) Let k be an algebraically closed field of characteristic $p > 0$. Show that the morphism $f : A^1_k \to A^1_k$ defined by $x \mapsto x^p$ is a bijective morphism but not an isomorphism. Where is the derivative of f zero? Is $f : A^1 \to A^1$ birational?

(6) Define the *support* of a coherent sheaf F on a scheme X to be the intersection of all closed subsets $S \subset X$ such that $F|_{X-S} = 0$. Show that $\operatorname{supp}(F)$ is itself a closed subset such that $F|_{X-\operatorname{supp}(F)} = 0$. Show that for X affine, $\operatorname{supp}(F)$ is the closed subset of X corresponding to an ideal in $O(X)$, namely the *annihilator* of the $O(X)$-module of sections $F(X)$:

$$\operatorname{ann} H^0(X, F) = \{f \in O(X) : fs = 0 \in H^0(X, F) \text{ for all global sections } s \in F(X)\}.$$

This is sometimes a useful way to check that two sheaves F_1, F_2 on X are not isomorphic: if they have different supports in X, they are not isomorphic.