Homework 3 for Math 214B Algebraic Geometry

Burt Totaro

Spring 2014, UCLA

Due on Monday, June 2.

In the following problems, varieties are over an algebraically closed field k, unless stated otherwise. A "curve of genus g" is usually understood to be smooth and projective over k.

(1) Let k be an algebraically closed field of characteristic $p > 0$. Show that the morphism $f : A^1_k \to A^1_k$ defined by $x \mapsto x^p$ is a bijective morphism but not an isomorphism. Where is the derivative of f zero? Is $f : A^1 \to A^1$ birational?

(2) Show that any curve of genus zero is isomorphic to $\mathbf{P}^1$.

(3) Show that any curve X of genus 1 can be written as a degree-2 ramified covering of $\mathbf{P}^1$ (meaning that there is a morphism $X \to \mathbf{P}^1$ of degree 2). Curves of genus 1 are called elliptic curves. Show that an elliptic curve is not rational (that is, it is not birational to $\mathbf{P}^1$).

(4) For an effective divisor D on a curve X of genus g, show that $h^0(X, O(D)) \leq \deg(D) + 1$. Show that equality holds if and only if $D = 0$ or $g = 0$.

(5) Let X be a smooth projective curve, p a point in X. Show that there is a nonconstant rational function on X which is regular outside p.

(6) A curve X is called *hyperelliptic* if it has genus $g \geq 2$ and there is a morphism $X \to \mathbf{P}^1$ of degree 2.

(a) If X is a curve of genus 2, show that the canonical bundle K_X defines a morphism $X \to \mathbf{P}^1$ of degree 2. Thus every curve of genus 2 is hyperelliptic.

(b) For $g(X) \geq 2$, show that the canonical bundle K_X defines a morphism $X \to \mathbf{P}^{g-1}$. (The main point here is to check that K_X is basepoint-free.) If X is not hyperelliptic, show that the canonical bundle defines an embedding of X in $\mathbf{P}^{g-1}$, the *canonical embedding.*

(c) Compute the genus of a smooth plane quartic curve X ("quartic" means degree 4), by describing the canonical bundle of X. Show that X is not hyperelliptic. (You may use that if a curve X of any genus $g \geq 2$ is hyperelliptic, then the canonical map $X \to \mathbf{P}^{g-1}$ is a double cover of its image, which is a rational normal curve.)

(7) Show that any elliptic curve X can be embedded as a smooth curve of degree d in $\mathbf{P}^{d-1}$ for any $d \geq 3$. Show that a transverse intersection of two smooth quadrics in $\mathbf{P}^3$ is indeed an elliptic curve of degree 4. But show that an elliptic curve of degree d in $\mathbf{P}^{d-1}$ is not a complete intersection for $d \geq 5$. (Hint: from Homework 2, you know the canonical bundle of any smooth complete intersection curve in any $\mathbf{P}^n$.)