Homework 1 for Math 214B Algebraic Geometry

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Spring 2014, UCLA

Due on Monday, April 21.

(1) Let X be a hypersurface of degree d in projective space $\mathbf{P}^n$ over a field k. Compute the cohomology groups $H^i(X, O_X)$ for all i. Deduce that X is not isomorphic to $\mathbf{P}^{n-1}$ for d large enough; what range of d do you get?

(2) Let X be an affine scheme of finite type over a field. We know that $H^i(X, E) = 0$ for every quasi-coherent sheaf E on X and every $i > 0$. Does this vanishing hold for every sheaf of O_X-modules, not necessarily quasi-coherent?

(3) Let $\mathcal{A}$ be an abelian category. Show (from the definition) that $\mathcal{A}$ has an initial object, which we call 0. Show that 0 is also a terminal object. Show that the coproduct $A \oplus B$ of two objects in $\mathcal{A}$ (which exists by definition of an abelian category) is also the product $A \times B$. (In additive categories such as $\mathcal{A}$, coproducts are usually called direct sums.)

Define a "subobject" of an object X in an abelian category as an equivalence class of monomorphisms to X. (Your definition should agree with the usual notion of a subgroup of an abelian group, for example.) Define what it means for one subobject of X to be contained in another. Show that the subobjects of X form a partially ordered class. (In general, it may not be a set.)

(4) Let X be the affine plane A^2 over a field k, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by affine open subsets, show that $H^1(U, O)$ is isomorphic to the k-vector space with basis $\{x^i y^j : i, j < 0\}$. In particular, it is a k-vector space of infinite dimension. Use this calculation to show that the scheme U is not affine.

(5) For a sheaf of abelian groups E on a topological space X, Cech cohomology is in general different from the "true" sheaf cohomology which we defined. But the natural map $H^i_{\text{Cech}}(X, E) \to H^i(X, E)$ is an isomorphism for $i \leq 1$, by exercise III.4.4 in Hartshorne (which you can use without proof). (Here Cech cohomology is defined by taking a direct limit over all open coverings.)

Deduce that for any ringed space (X, O_X), the group $\mathrm{Pic}(X)$ of isomorphism classes of line bundles (also called invertible sheaves in Hartshorne's section II.6) is isomorphic to $H^1(X, O_X^*)$, where O_X^* denotes the sheaf whose group of sections on an open set U is the group of invertible elements in the ring $O_X(U)$. [Hint: For any line bundle L on X, cover X by open sets U_i on which L is trivial, and fix isomorphisms $\varphi_i : O_{U_i} \to L|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $O_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $H^1(\mathcal{U}, \mathcal{O}_{\mathcal{X}}^*)$ (Cech cohomology with respect to the given covering $\mathcal{U}$). Now use exercise III.4.4 as mentioned above.]

(6) Let X be a smooth scheme over a field k, Y a smooth closed subscheme, and $I = I_{Y/X}$ the ideal sheaf of Y in X. Let $2Y$ be the closed subscheme defined by the sheaf of ideals I^2. The sheaf I/I^2 on X is the direct image of a vector bundle $N_{Y/X}^*$ on Y (the dual of the normal bundle of Y in X). Show that there is an exact sequence of sheaves of abelian groups on X.

$$0 \to N_{Y/X}^* \to O_{2Y}^* \to O_Y^* \to 0,$$

where O_Y^* denotes the sheaf of (multiplicative) groups of units in the sheaf of rings O_Y, $N_{Y/X}^* = I/I^2$ has its usual (additive) group structure, and the map $I/I^2 \to O_{2Y}^*$ is given by $a \mapsto 1 + a$. Conclude that there is an exact sequence of abelian groups

$$\cdots \to H^1(Y, N_{Y/X}^*) \to \mathrm{Pic}(2Y) \to \mathrm{Pic}(Y) \to H^2(Y, N_{Y/X}^*) \to \cdots .$$

(7) Let X be a noetherian separated scheme. Define the *cohomological dimension* of X, denoted $\mathrm{cd}(X)$, to be the least integer n such that $H^i(X, F) = 0$ for all quasi-coherent sheaves F and all $i > n$. For example, Serre's Theorem III.3.7 in Hartshorne says that $\mathrm{cd}(X) = 0$ if and only if X is affine. Grothendieck's Theorem III.2.7 implies that $\mathrm{cd}(X) \leq \dim(X)$.

(a) In the definition of $\mathrm{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use exercise II.5.15 and Prop. III.2.9.

(b) If X is quasi-projective over a field k, then it is even sufficient to consider vector bundles on X. Use Cor. II.5.18.

(c) Suppose that X has a covering by $r + 1$ open affine subsets. Use Cech cohomology to show that $\mathrm{cd}(X) \leq r$.

(d) If X is quasi-projective scheme of dimension r over a field k, show that X can be covered by $r + 1$ open affine subsets. Conclude (independent of Grothendieck's theorem) that $\mathrm{cd}(X) \leq \dim(X)$.

(e) Let Y be a set-theoretic complete intersection (exercise I.2.17) of codimension r in $X = \mathbf{P}_k^n$. Show that $\mathrm{cd}(X - Y) \leq r - 1$.

(8) Let $X = \mathrm{Spec}\, k[x_1, x_2, x_3, x_4]$ be affine 4-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y} \subset \mathbf{P}_k^4$ is also not a set-theoretic complete intersection. [Hint: Use an affine analogue of problem 7(e) above. Then show that $H^2(X - Y, O_X) \neq 0$, by using exercises III.2.3 (cohomology with support) and III.2.4 (Mayer-Vietoris).]