Homework 3 for Math 214A Algebraic Geometry

Burt Totaro

February 19, 2020

Due Monday, Mar. 9.

(1) Show that the conic $xy = z^2$ in $\mathbf{P}^2$ over any field k is isomorphic to $\mathbf{P}^1$. Show that every conic (irreducible curve of degree 2 in $\mathbf{P}^2$) over an algebraically closed field of characteristic not 2 can be moved to $xy = z^2$ by some automorphism of $\mathbf{P}^2$.

(2) Show that any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X such that Y is irreducible in the subspace topology, then the closure $\overline{Y}$ is also irreducible.

(3) *Linear subspaces of* $\mathbf{P}^n$. Let k be a field. A hypersurface in $\mathbf{P}^n$ over k of degree 1 (that is, defined by a homogeneous polynomial of degree 1) is called a *hyperplane*. A nonempty intersection of hyperplanes is called a *linear subspace* of $\mathbf{P}^n$.

(a) If Y is a linear subspace of dimension r in $\mathbf{P}^n$, show that Y is isomorphic to $\mathbf{P}^r$.

(b) Let Y, Z be linear subspaces of dimension r and s in $\mathbf{P}^n$. If $r + s - n \geq 0$, then $Y \cap Z \neq \emptyset$. Moreover, $Y \cap Z$ is a linear subspace of dimension at least $r+s-n$ in $\mathbf{P}^n$. (Think of $A^{n+1}(k)$ as a vector space over k, and work with its subspaces.)

(4) *The Veronese embedding.* Let k be a field. For a given $n, d > 0$, let $M_0, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{d} - 1$. We define a morphism $\rho_d : \mathbf{P}^n \to \mathbf{P}^N$ over k by sending a point $P = [a_0, \ldots, a_n]$ to the point $[M_0(a), \ldots, M_N(a)]$ obtained by substituting the numbers a_i in the monomials M_j. This is called the dth *Veronese embedding*, or the *d-uple embedding*, of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, when $n = 1$ and $d = 2$, this is the embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ as a conic.

(a) Show that the dth Veronese map of $\mathbf{P}^1$, $\rho_d : [u, v] \mapsto [u^d, u^{d-1}v, \ldots, v^d]$, is a morphism from $\mathbf{P}^1$ to $\mathbf{P}^d$. Show that the image is closed in $\mathbf{P}^d$. Show that ρ_d is an isomorphism from $\mathbf{P}^1$ to this closed subset, called a *rational normal curve* in $\mathbf{P}^d$. (These results hold for the Veronese embeddings of $\mathbf{P}^n$ for any n, but it takes longer to write out the proofs in general.)

(b) Show that the rational normal curve in $\mathbf{P}^3$ (called the *twisted cubic curve*) is the projective closure of the affine curve $\{(t, t^2, t^3)\} \subset A^3$, in some coordinates.

(5) Show that any two curves (varieties of dimension 1) over a field k are homeomorphic in the Zariski topology. (This shows that the Zariski topology, by itself, carries little information. It must be combined with the sheaf of regular functions to be useful.) On the other hand, show that the affine plane A^2 is not homeomorphic to any curve over k in the Zariski topology.

(6) If $X \subset \mathbf{P}^n$ over a field k is any hypersurface, show that $\mathbf{P}^n - X$ is affine. [Hint: Let X have degree d. Then consider the dth Veronese embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$. What does $\mathbf{P}^n - X$ look like in terms of that embedding?]

(7) Show that any two curves in $\mathbf{P}^2$ over a field k have nonempty intersection. More generally, show that if $X \subset \mathbf{P}^n$ is a projective variety of dimension ≥ 1, and if $D \subset \mathbf{P}^n$ is a hypersurface, then $X \cap D \neq \emptyset$. [Hint: use that $\mathbf{P}^n - D$ is affine, by the previous problem. Depending on your approach, you may want to prove the result first when k is algebraically closed and then deduce the general statement.]

(8) (a) Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial. Suppose that the subset of $\{f = 0\} \subset \mathbf{P}^n$ defined by $\partial f/\partial x_0 = 0, \ldots, \partial f/\partial x_n = 0$ is empty (so $\{f = 0\}$ is a smooth hypersurface in $\mathbf{P}^n$). If $n \geq 2$, show that the polynomial f is irreducible over k and hence the hypersurface $\{f = 0\} \subset \mathbf{P}^n$ is irreducible. [Hint: use the previous problem.]

(b) Let k be an algebraically closed field of characteristic zero. For any $d \geq 1$, show that the Fermat curve $x^d + y^d + z^d = 0$ in $\mathbf{P}^2_k$ is smooth and irreducible.

(9) (a) Let X and Y be varieties over a field k. If k is algebraically closed, show that $X \times Y$ (meaning $X \times_k Y$) is a variety over k. Give an example to show that this can fail for k not algebraically closed.

(b) Show that for schemes X and Y of finite type over a field k, $\dim(X \times Y) = \dim(X) + \dim(Y)$.