Homework 1 for Math 214A Algebraic Geometry

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Due Wednesday, Jan. 22. Varieties are over an algebraically closed field k, unless stated otherwise.

(1) Let X be the closed algebraic subset of the affine plane A^2 over k defined by $x^2 + y^2 = 1$ and $x = 1$. What is the ideal $I(X)$ in $k[x, y]$ of functions vanishing on X?

(2) Let X be the closed subset of A^3 over k defined by $x^2 + y^2 + z^2 = 0$. Determine the ideal $I(X)$ when the characteristic of k is 2. Determine $I(X)$ when the characteristic of k is not 2.

(3) Let X be an affine algebraic set and $X = \bigcup U_\alpha$ a cover of X by open subsets U_α. Prove that X is the union of finitely many of the subsets U_α.

(4) Every affine algebraic set X is the union of finitely many varieties; that is, $X = X_1 \cup X_2 \cup \cdots \cup X_m$ where X_i are irreducible closed subsets of X. If X_i is not contained in X_j for any $i \neq j$, we call X_i the irreducible components of X. Now decompose $X = \{(x, y, z) \in A^3 : x^2 = yz, xz = x\}$ into its irreducible components.

(5) Let $f : A^1_k \to A^1_k$ be an isomorphism. Prove that f is given by a linear polynomial.

(6) Show that the varieties A^1 and $A^1 - \{0\}$ are not isomorphic. Likewise for A^2 and $A^2 - \{0\}$, which is a bit different.

(7) Let $X = \{(x, y) \in A^2 : x^2 = y^3\}$. Define a bijective morphism from A^1 to X. Show that this is not an isomorphism. In fact, show that A^1 and X are not isomorphic.

(8) Let k be an algebraically closed field of characteristic zero. Find the singular points of the affine curve $xy + x^3 + y^3 = 0$ over k.