7.3 Convex Games. If in Theorem 7.2, we add the assumption that the payoff function $A(x,y)$ is convex in y for all x or concave in x for all y, then we can say a lot more about the optimal strategies of the players. Here is a one-sided version that complements Theorem 7.3.

Theorem 7.4. *Let (X, Y, A) be a game with X arbitrary, Y a compact convex subset of $\mathbb{R}^n$, and $A(x, \boldsymbol{y})$ bounded above. If $A(x, \boldsymbol{y})$ is a convex function of $\boldsymbol{y} \in Y$ for all $x \in X$, then the game has a value and Player II has an optimal pure strategy. Moreover, Player I has an ϵ-optimal strategy that is a mixture of at most $n+1$ pure strategies.*

The game is solved by a method similar to solving m by 2 games. The optimal strategy of Player II has a simple description. Let $g(\boldsymbol{y}) = \sup_x A(x, \boldsymbol{y})$ be the upper envelope. Then $g(\boldsymbol{y})$ is finite since A is bounded above, and convex since the supremum of any set of convex functions is convex. Therefore, there exists a point $\boldsymbol{y}^*$ at which $g(\boldsymbol{y})$ takes on its minimum value, so that

$$A(x, \boldsymbol{y}^*) \le \max_x A(x, \boldsymbol{y}^*) = g(\boldsymbol{y}^*) \qquad \text{for all } x \in X.$$

Any such point is an optimal pure strategy for Player II. Player II can guarantee she will lose no more than $g(\boldsymbol{y}^*)$. Player I's optimal strategy is more complex to describe in general; it gives weight only to points that play a role in the upper envelope at the point $\boldsymbol{y}^*$. These are points x such that $A(x, \boldsymbol{y})$ is tangent (or nearly tangent if only ϵ-optimal strategies exist) to the surface $g(\boldsymbol{y})$ at $\boldsymbol{y}^*$. It is best to consider examples.

Example 1. Estimation. Player I chooses a point $x \in X = [0,1]$, and Player II tries to choose a point $y \in Y = [0,1]$ close to x. Player II loses the square of the distance from x to y: $A(x,y) = (x-y)^2$. This is a convex function of $y \in [0,1]$ for all $x \in X$. Any $A(x,y)$ is bounded above by either $A(0,y)$ or $A(1,y)$ so the upper envelope is $g(y) = \max\{A(0,y), A(1,y)\} = \max\{y^2, (1-y)^2\}$. This is minimized at $y^* = 1/2$. If Player II uses y^*, she is guaranteed to lose no more than $g(y^*) = 1/4$.

Since $x = 0$ and $x = 1$ are the only two pure strategies influencing the upper envelope, and since y^2 and $(1-y)^2$ have slopes at y^* that are equal in absolute value but opposite in sign, Player I should mix 0 and 1 with equal probability. This mixed strategy has convex payoff $(1/2)(A(0,y) + A(1,y))$ with slope zero at y^*. Player I is guaranteed winning at least 1/4, so $v = 1/4$ is the value of the game. The pure strategy y^* is optimal for Player II and the mixed strategy, 0 with probability 1/2 and 1 with probability 1/2, is optimal for Player I. In this example, $n = 1$, and Player I's optimal strategy mixes $2 = n+1$ points. ■

Theorem 7.4 may also be stated with the roles of the players reversed. If Y is arbitrary, and if X is a compact subset of $\mathbb{R}^m$ and if $A(\boldsymbol{x}, y)$ is bounded below and concave in $\boldsymbol{x} \in X$ for all $y \in Y$, then Player I has an optimal pure strategy, and Player II has an ϵ-optimal strategy mixing at most $m+1$ pure strategies. It may also happen that $A(x,y)$ is concave in x for all y, and convex in y for all x. In that case, both players have optimal pure strategies as in the following example.

Example 2. A Convex-Concave Game. Suppose $X = Y = [0,1]$, and $A(x,y) = -2x^2 + 4xy + y^2 - 2x - 3y + 1$. The payoff is convex in y and concave in x. Both players have pure optimal strategies, say x_0 and y_0. If Player II uses y_0, then $A(x, y_0)$ must be maximized by x_0. To find $\max_{x\in[0.1]} A(x, y_0)$ we take a derivative with respect to x: $\frac{\partial}{\partial x}A(x, y_0) = -4x + 4y_0 - 2$. So

$$x_0 = \begin{cases} y_0 - (1/2) & \text{if } y_0 > 1/2 \\ 0 & \text{if } y_0 \le 1/2 \end{cases}$$

Similarly, if Player I uses x_0, then $A(x_0, y)$ is minimized by y_0. Since $\frac{\partial}{\partial y}A(x_0, y) = 4x_0 + 2y - 3$, we have

$$y_0 = \begin{cases} 1 & \text{if } x_0 \le 1/4 \\ (1/2)(3 - 4x_0) & \text{if } 1/4 \le x_0 \le 3/4 \\ 0 & \text{if } x_0 \ge 3/4. \end{cases}$$

These two equations are satisfied only if $x_0 = y_0 - (1/2)$ and $y_0 = (1/2)(3 - 4x_0)$. It is then easily found that $x_0 = 1/3$ and $y_0 = 5/6$. The value is $A(x_0, y_0) = -7/12$.

It may be easier here to find the saddle-point of the surface, $z = -2x^2 + 4xy + y^2 - 2x - 3y + 1$, and if the saddle-point is in the unit square, then that is the solution. But the method used here shows what must be done in general. ∎

Exercise 5. Find optimal strategies and the value of the following games.

(a) $X = Y = [0, 1]$ and $A(x, y) = \begin{cases} (x - y)^2 & \text{if } x \le y \\ 2(x - y)^2 & \text{if } x \ge y. \end{cases}$ (Underestimation is the more serious error of Player II.)

(b) $X = Y = [0, 1]$ and $A(x, y) = xe^{-y} + (1 - x)y$.

Solutions. 5. (a) The upper envelope is $\max\{A(0, y), A(1, y)\} = \max\{y^2, 2(1 - y)^2\}$. This has a minimum when $y^2 = 2(1 - y)^2$. This reduces to $y^2 - 4y + 2 = 0$ whose solution in $[0, 1]$ is $y_0 = 2 - \sqrt{2} = .586\cdots$. The slope of $A(0, y)$ and that of $A(1, y)$ at $y = y_0$ is proportional to $2y_0 : -4 + 2y_0$ which reduces to $2 - \sqrt{2} : \sqrt{2}$. So Player I's optimal strategy is mix $x = 0$ and $x = 1$ with probabilities $(2 - \sqrt{2})/2$ and $\sqrt{2}/2$, respectively. Numerically this is $(.293\cdots, .707\cdots)$.

(b) This is a convex-concave game so both player have optimal pure strategies. If y_0 is an optimal pure strategy for Player II, then x_0 must maximize $A(x, y_0)$. As a function of x this is a line of slope $e^{-y_0} - y_0$. So

$$x_0 = \begin{cases} 0 & \text{if } e^{-y_0} < y_0 \\ \text{any} & \text{if } e^{-y_0} = y_0 \\ 1 & \text{if } e^{-y_0} > y_0 \end{cases}$$

We are bound to have a solution to this equation if $e^{-y_0} = y_0$. So $y_0 = .5671\cdots$. But y must minimize $A(x_0, y)$, whose derivative, $-x_0e^{-y} + 1 - x_0$ must be zero at y_0. This gives $x_0(e^{-y_0} + 1) = 1$. Since $e^{-y_0} = y_0$, we have $x_0 = 1/(1 + y_0) = .6381\cdots$.