Exercise 6. The Wallet Game. Two players each put a random amount with mean one into their wallets. The player whose wallet contains the smaller amount wins the larger amount from the opponent.

Carroll, Jones and Rykken (2001) show that this game does not have a value. But suppose we restrict the players to putting at most some amount b in their wallets. Here is the game:

Player I, resp. Player II, chooses a distribution F, resp. G, on the interval $[0, b]$ with mean 1, where $b > 1$. Then independent random variables, X from F and Y from G, are chosen. If $X < Y$, Player I wins Y from Player II. If $X > Y$, Player II wins X from Player I, and if $X = Y$, there is no payoff. So the payoff function is

$$\begin{aligned} A(F, G) &= \mathrm{E}(Y\mathrm{I}(X < Y) - X\mathrm{I}(X > Y)) \\ &= \mathrm{E}((Y + X)\mathrm{I}(X < Y)) - 1 + \mathrm{E}(X\mathrm{I}(X = Y)). \end{aligned} \tag{1}$$

The game is symmetric, so if the value exists, the value is zero, and the players have the same optimal strategies. Find an optimal strategy for the players. Hint: Search among distributions F having a density f on the interval (a, b) for some $a < 1$. Note that the last term on the right of Equation (1) disappears for such distributions.

Solution. Let us assume that the "good" strategies are those that give all mass to some interval, $[a, b]$, with $0 < a < 1 < b$, and let us search for a strategy F with a density $f(x)$ for which $A(F, G)$ is constant over good strategies, G. Let $\Phi(y) = \int_a^y xf(x)\,dx$. Then

$$A(F, G) = \int_a^b \int_a^y (y + x)f(x)\,dx\,dG(y) - 1 = \int_a^b [yF(y) + \Phi(y)]\,dG(y) - 1 \tag{2}$$

Since $\int_a^b dG(y) = 1$ and $\int_a^b y\,dG(y) = 1$, the payoff (2) will be constant for all good G, provided $yF(y) + \Phi(y)$ is linear in $y \in (a, b)$, for some α. This means the second derivative of $yF(y) + \Phi(y)$ must be zero:

$$3yf(y) + 2f'(y) = 0 \tag{3}$$

whose solution is

$$f(y) = cy^{-3/2} \qquad \text{for } y \in (a, b), \tag{4}$$

for some constant $c > 0$. We have

$$F(y) = 2c[a^{-1/2} - y^{-1/2}] \quad \text{and} \quad \Phi(y) = 2c[y^{1/2} - a^{1/2}] \qquad \text{for } a < y < b \tag{5}$$

For f to be a density, the constant c must satisfy

$$2c(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}) = 1. \tag{6}$$

For $\mathrm{E}(X) = 1$, we must have

$$2c(\sqrt{b} - \sqrt{a}) = 1. \tag{7}$$

We see from (6) and (7) that $b = 1/a$, and $c = \sqrt{b}/(2(b-1))$. Therefore from (5),

$$yF(y) + \Phi(y) = \frac{2c}{\sqrt{a}}(y-a) = \frac{b}{b-1}(y - \frac{1}{b}) \tag{8}$$

The value of this game is zero. The distribution with density

$$f(y) = \sqrt{b}/(2(b-1))y^{-3/2} \qquad \text{for } 1/b < y < b \tag{9}$$

is an optimal pure strategy for both players.

Proof. Suppose Player I uses $f(x)$ of (9). Since there can be no ties, we must show that if Player II uses any distribution function $G(y)$ on $[0, b]$ having mean 1, then $\mathrm{E}((Y+X)\mathrm{I}(X < Y)) \geq 1$. But,

$$\begin{aligned}
\mathrm{E}((Y+X)\mathrm{I}(X<Y)) &= \int_{1/b}^{b}\int_{1/b}^{y}(y+x)f(x)\,dx\,dG(y) = \int_{1/b}^{b}[yF(y)+\Phi(y)]\,dG(y) \\
&= \int_{1/b}^{b}\frac{b}{b-1}(y-\frac{1}{b})\,dG(y) \\
&\geq \int_{0}^{b}\frac{b}{b-1}(y-\frac{1}{b})\,dG(y) = \frac{b}{b-1}(1-\frac{1}{b}) = 1
\end{aligned}$$

This also shows that if Player II puts any mass below $1/b$, then Player I's expected payoff is positive.

Reference.

Maureen T. Carroll, Michael A. Jones, Elyn K. Rykken (2001) "The Wallet Paradox Revisited", *Mathematics Magazine* **74**, 378-383.