

On the Borel and von Neumann Poker Models

Chris Ferguson, Bright Trading, Westwood, California

Thomas S. Ferguson, University of California, Los Angeles

1. Introduction and Summary.

The study of two-person zero-sum poker models with independent uniform hands goes back to Borel and von Neumann. Borel discusses a form of poker in Chapter 5, “Le jeu de poker” of his 1938 book, *Applications aux Jeux des Hazard*. Von Neumann presents his analysis of a similar form of poker in the seminal book on game theory — *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1944). Section 19 of the book is devoted to certain mathematical models of poker, with both discrete and continuous hands, and with both simultaneous bets and alternating bets. Extensions of the model of Borel may be found in the work of Bellman and Blackwell (1949), Bellman (1952), and Karlin and Restrepo (1957).

In these models, Player I is dealt a random hand $X \in [0, 1]$ where X has a uniform distribution over the interval $[0, 1]$; the prior probability that X is in any subinterval of $[0, 1]$ is the length of the subinterval. Similarly, Player II independently receives a random hand, Y , according to a uniform distribution on $[0, 1]$. Throughout the play, both players know the value of their own hand, but not that of the opponent. The structure of the betting in the two models is the same. Each player antes one unit. Player I first decides whether or not to bet. If Player I bets, then Player II decides whether to call or to fold. If Player II folds, Player I wins one unit (the ante) from Player II. If Player II calls, the hands are compared and the player with the higher hand wins an amount $B + 1$ from the

opponent, where $B > 0$ represents the amount of the bet. The two models differ only in what happens if Player I chooses not to bet. In the Borel model, Player I folds: Player I loses one unit to Player II. In the von Neumann model, Player I checks: the hands are compared and the player with the higher hand wins 1 from the opponent.

These models are described in more detail in Sections 2 and 3. The values and optimal strategies as discovered by Borel and von Neumann are stated. Because the derivation of the optimal strategies as given by Borel and von Neumann are involved, we give a derivation using a standard contemporary method, called here the principle of indifference. The notions of bluffing, mistakes and admissible strategies are explained. A particular strategy, σ , for a player is called a *mistake* if there exists an optimal strategy for the opponent which when used against σ gives the opponent an expected payoff better than the value of the game. A strategy is said to be *admissible* for a player if no other strategy for that player does better against one strategy of the opponent without doing worse against some other strategy of the opponent. An admissible optimal strategy is useful because it takes advantage of mistakes an opponent may make. We show that in both models, both players have unique admissible optimal strategies. In the model of Borel, Player I bluffs with the best of the hands he does not bet with. This is the best way to take advantage of mistakes of Player II. In the more realistic model of von Neumann, Player I bluffs with his worst hands. It is a mistake for him to do otherwise. This is a phenomenon known to professional gamblers, that in bluffing, one should use one's worst hands. Finally it is noted that in the von Neumann model, there is an optimal bet size. If Player I gets to choose the bet size, B , it should be chosen to be $B = 2$; that is, the optimal bet is the size of the pot.

After the dealing of the hands, all actions that the players take are announced. Thus, except for the dealing of the hands at the start of the game, this would be a game of perfect information. Games of this sort, where, after an initial random move giving secret information to the players, the game is played with no further introduction of hidden information, are called games of *almost perfect information*. Techniques for solving such games have been studied by Ponsard (1975) and applied to a poker model by Sorin and Ponsard (1980). It is convenient to study the action part of games of almost complete

information by what we call the betting tree. This is distinct from the Kuhn tree in that it neglects the information sets that may arise from the initial distribution of hands. Examples illustrate this concept.

The model of Borel is a poor model of poker because of the feature that a player must fold if he does not bet. The model of von Neumann is somewhat better in that it corrects this point. It is therefore interesting and somewhat surprising that most of the subsequent literature on the subject extend the model of Borel rather than that of von Neumann. In particular, the models of Bellman and Blackwell and of Karlin and Restrepo, allowing Player I a choice of bet sizes, extend the model of Borel. It might seem that such a model would contain the model of von Neumann also by allowing one of the bet sizes to be zero. However, in both cases, lower bounds are placed on the bet sizes that disallow this possibility. Therefore, it is of interest to remove this restriction on the bet sizes and thus to create a bridge between the models of Borel and von Neumann. After the discussion of the basic models in Sections 2 and 3, the bridge is built in Section 4.

It is generally assumed that X and Y are *independent* random variables; that is, learning the value of his own hand gives a player no information about the hand of his opponent. This assumption would not be satisfied if the players were dealt distinct hands from a finite deck, so it is important to weaken this assumption. Some work has been done by Sakaguchi and Sakai (1981) for the Borel model using the Farlie-Gumbel-Morgenstern (FGM) distributions. This is the family of distributions with a bilinear density on $[0, 1]^2$ for which the marginal distributions of X and Y are uniform. There is one parameter that controls dependence. Sakaguchi and Sakai find the value and optimal strategies in the case of a negative dependence of the hands (i.e. a high hand for one player tends to go with a low hand of the opponent). However, the maximum and minimum correlations that can be obtained in the FGM family are $+1/3$ and $-1/3$ so it is important to obtain more general results. In Section 5, the players are allowed general dependent distributions, with possibly different marginals, in order to see just what type of negative dependence is required for the solution of Sakaguchi and Sakai to be valid. A similar treatment is given to the von Neumann model. The general case including the positive dependent case (when higher hands tend to occur together) remains completely open.

2. La Relance.

In his book, Borel introduces a model of poker he calls “la relance”. Each player contributes an ante of 1 unit into the pot, and then they receive independent uniform hands on the interval $[0, 1]$. Player I acts first either by folding and thus conceding the pot to Player II, or by betting a prescribed amount $B > 0$ which he adds to the pot. If Player I bets, then Player II acts either by folding and thus conceding the pot to Player I, or by calling and adding B to the pot. If Player II calls the bet of Player I, the hands are compared and the player with the higher hand wins the entire pot. That is, if $X > Y$ then Player I wins the pot; if $X < Y$ then Player II wins the pot. We do not have to consider the case $X = Y$ since this occurs with probability 0.

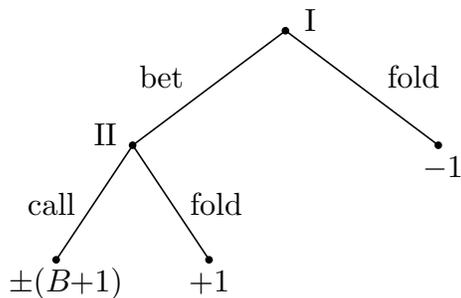


Fig. 1. The betting tree for La Relance

The betting tree for La Relance is displayed in Figure 1. In this diagram, the value at the end of each line of play indicate the winnings of Player I. The plus-or-minus sign indicates that the hands are compared, and the higher hand wins the amount $B + 1$.

Theorem 1. *The value of La Relance is*

$$V(B) = -\frac{B^2}{(B+2)^2}. \quad (1)$$

The unique optimal strategy for Player II is to call if $Y > c$ and to fold otherwise, where

$$c = B/(B+2). \quad (2)$$

An optimal strategy for Player I is to bet if $X > c^2$ and to fold otherwise.

The fact that the value is negative shows that the game favors Player II.

Here is a simple way to find the unique strategy for Player II using the principle of indifference. It is useful to consider the money already put into the pot as a sunk cost, belonging to neither player. This views the game as a constant-sum game, where the sum of the players' winnings is 2 whatever the outcome. This is a convenient way to view the pot. One may guess and it is easy to prove that the optimal strategy for Player II is of the form for some $c \in [0, 1]$, fold if $Y < c$ and call if $Y > c$. Player II chooses c to make I indifferent between betting and folding when I has some hand $X < c$. If I bets with such an X , he wins 2 (the pot) if II has $Y < c$ and loses B if II has $Y > c$. His expected winnings are in this case, $2c - B(1 - c)$. On the other hand, if I folds he wins nothing. He will be indifferent between betting and folding if $2c - B(1 - c) = 0$ from which (2) follows.

Player I's optimal strategy is not unique, but Borel finds all of them. These strategies are of the form: if $X > c$, bet; and if $X < c$, bet only a certain proportion of the time. We may find this proportion using the principle of indifference. Let π denote the proportion of time below c that Player I bets; then $P(X < c | \text{I bets}) = c\pi / (c\pi + (1 - c))$. Player I chooses π to make Player II indifferent between calling and folding when she has $Y = c$. If Player II calls with $Y = c$, she wins $B + 2$ with probability $P(X < c | \text{I bets})$ and loses B with probability $P(X > c | \text{I bets})$. Her expected winnings are $(B + 2)P(X < c | \text{I bets}) - BP(X > c | \text{I bets})$. If she folds, she wins nothing, so she is indifferent if $(B + 2)c\pi - B(1 - c) = 0$. Solving for π gives $\pi = 1 - c = 2/(B + 2)$.

Player I's optimal strategies allow him to do as he pleases provided the proportion of times he bets with hands $X < c$ is $1 - c$, and the proportion of times he folds these hands is c . For example, Player I may fold with his worst hands, i.e. with $X < c^2$, or he may fold with the best of his hands less than c , i.e. with $c - c^2 < X < c$, or he may, for all $0 < X < c$, simply toss a coin with probability c of heads and fold if the coin comes up heads. All such strategies are optimal. But we now point out that some optimal strategies are better than others.

Pot-Limit Poker, Bluffing, Mistakes and Admissible Strategies. When the size of the bet is restricted to be no larger than the size of the pot, the game is called *pot-limit poker*. In La Relance, suppose $B = 2$, the maximum allowable bet in pot-limit. Then $c = 1/2$; an optimal strategy for Player I is to bet if $X > 1/4$ and fold otherwise; the optimal strategy of Player II is to call if $Y > 1/2$. The value is $-1/4$. So Player II's expected gain is $1/4$ unit each time the game is played.

If I bets when $X < c$, he knows he will lose if called, assuming II is using an optimal strategy. Such a bet is called a *bluff*. In La Relance, it is necessary for I to bluff with probability $c - c^2$. Which of the hands below c he chooses to bluff with is immaterial as far as the value of the game is concerned. However, there is a secondary advantage to bluffing (betting) with the hands just below c , that is, with the hands from c^2 to c . Such a strategy takes maximum advantage of a mistake the other player may make.

A particular strategy σ for a player is called a *mistake* if there exists an optimal strategy for the opponent which when used against σ gives the opponent an expected payoff better than the value of the game. In La Relance, it is a mistake for Player II to call with some $Y < c$ or to fold with some $Y > c$. If II calls with some $Y < c$, then I can gain from the mistake most profitably if he bluffs only with his best hands below c .

A strategy is said to be *admissible* for a player if no other strategy for that player does better against one strategy of the opponent without doing worse against some other strategy of the opponent. *The strategy of betting if and only if $X > c^2$ is the unique admissible optimal strategy for Player I.* For this reason, we do not include the other optimal strategies of Player I in the statement of Theorem 1. Player I should not use them.

3. The von Neumann Model.

The model of von Neumann differs from the model of Borel in one small but significant respect. If Player I does not bet, he does not necessarily lose the pot. Instead the hands are immediately compared and the higher hand wins the pot. We say Player I checks rather than folds. This provides a better approximation to real poker and a clearer example of

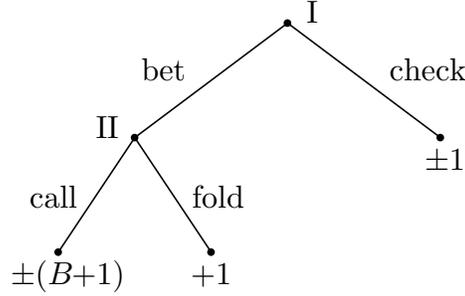
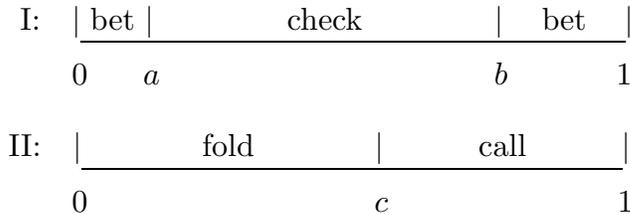


Fig 2. The betting tree for von Neumann's poker

the concept of bluffing in poker. The betting tree of von Neumann's poker is the same as Borel's except that the -1 payoff on the right branch is changed to ± 1 .

This time it is Player I that has a unique optimal strategy. It is of the form for some numbers a and b with $a < b$: bet if $X < a$ or if $X > b$, and check otherwise. Although there are many optimal strategies for Player II (and von Neumann finds all of them), there is a unique admissible one and it has the simple form: call if and only if $Y > c$ for some number c . It turns out that $0 < a < c < b < 1$.



The results of von Neumann contain the following theorem.

Theorem 2. *The value of von Neumann's poker is*

$$V(B) = \frac{B}{(B+1)(B+4)}. \tag{3}$$

An optimal strategy for Player I is to check if $a < X < b$ and to bet otherwise, where

$$a = \frac{B}{(B+1)(B+4)} \quad \text{and} \quad b = \frac{B^2 + 4B + 2}{(B+1)(B+4)}. \tag{4}$$

An optimal strategy for Player II is to call if $Y > c$ and to fold otherwise, where

$$c = \frac{B(B+3)}{(B+1)(B+4)}. \tag{5}$$

Note that the game favors Player I. For pot-limit poker where $B = 2$, we have $a = 1/9$, $b = 7/9$, and $c = 5/9$, and the value is $V(2) = 1/9$.

It is interesting to note that there is an optimal bet size for Player I. It may be found by setting the derivative of $V(B)$ to zero and solving the resulting equation for B . It is $B = 2$. In other words, the optimal bet size is the size of the pot, exactly pot-limit poker! (This assumes that the bet size for the game is fixed before play begins. For a model in which Player I may choose the bet size after he observes X , see the paper of Newman (1959).)

The region $x < a$ is the region in which Player I bluffs. It is noteworthy that *Player I must bluff with his worst hands*, and not with his moderate hands. It is a mistake for Player I to do otherwise. Here is a rough explanation of this somewhat counterintuitive feature. Hands below c may be used for bluffing or checking. For bluffing it doesn't matter much which hands are used; one expects to lose them if called. For checking though it certainly matters; one is better off checking with the better hands.

The other optimal strategies for Player II are those that allow folding or calling with a hand between a and b provided the average probability of calling in this region is $(b - c)/(b - a)$. However, to take advantage of any mistake Player I may make by bluffing with other than his poorest hands, Player II must call with only the best hands in this region; The only admissible optimal strategy of Player II is that stated in Theorem 2.

Since the derivation of this result as given by von Neumann involves solving a discrete version of the problem and passing to the limit, it might be worthwhile to see how easily the result follows using the principle of indifference.

Let us apply the principle of indifference to find the optimal values of a , b and c . This will lead to three equations in three unknowns, known as the *indifference equations*. First, Player II should be indifferent between folding and calling with a hand $Y = c$. Again we use the gambler's point of view of the game as a constant sum game, where winning what is already in the pot is considered as a bonus. If II folds, she wins zero. If she calls with

$Y = c$, she wins $(B + 2)$ if $X < a$ and loses B if $X > b$. Equating her expected winnings gives the first indifference equation,

$$(B + 2)a - B(1 - b) = 0. \tag{6}$$

Second, Player I should be indifferent between checking and betting with $X = a$. If he checks with $X = a$, he wins 2 if $Y < a$, and wins nothing otherwise, for an expected return of $2a$. If he bets, he wins 2 if $Y < c$ and loses B if $y > c$, for an expected return of $2c - B(1 - c)$. Equating these gives the second indifference equation,

$$2c - B(1 - c) = 2a. \tag{7}$$

Third, Player I should be indifferent between checking and betting with $X = b$. If he checks, he wins 2 if $Y < b$. If he bets, he wins 2 if $Y < c$ and wins $B + 2$ if $c < Y < b$, and loses B if $Y > b$, for an expected return of $2c + (B + 2)(b - c) - B(1 - b)$. This gives the third indifference equation,

$$2c + (B + 2)(b - c) - B(1 - b) = 2b,$$

which reduces to

$$2b - c = 1. \tag{8}$$

The optimal values of a , b and c can be found by solving equations (6) (7) and (8) in terms of B . The solution is as given in (4) and (5), and these give rise to the value in (3).

4. A Bridge from Borel to von Neumann

An extension of the model of Borel is found in the work of Bellman and Blackwell (1949) and Bellman (1952). In this extension, Player I is allowed to choose between two sizes of bets; that is, after observing his hand X , Player I may fold, or bet B_1 , or bet B_2 , where $0 \leq B_1 \leq B_2$. (See the betting tree below.) In the case $B_1 = B_2$, this is exactly Borel's model. In the case $B_1 = 0$, this is equivalent to the model of von Neumann, since Player I would never fold if he can bet 0, which here is equivalent to checking. Therefore,

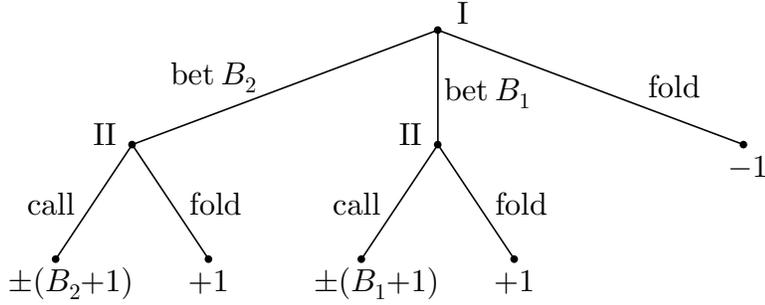


Fig 3. The betting tree for the Bellman-Blackwell Model

such a model may be used to provide a bridge between the model of Borel and the model of von Neumann.

However, in the solution to this model by Bellman and Blackwell, and in the subsequent extension by Karlin and Restrepo (1957) to an arbitrary number of bet sizes, lower bounds are placed on B_1 so that it is not allowed to approach zero and the solution does not contain the von Neumann solution. We first review the Bellman-Blackwell solution, and then fill in the extension to the von Neumann model.

Bellman and Blackwell find optimal strategies for the two players of the following form based on five numbers, $b_1 < b_2 < b_3$, $m_H > 0$, and $m_L > 0$.

For Player I:

- (a) bet high if $X > b_3$.
- (b) bet low if $b_1 < X < b_3$.
- (c) If $X < b_1$, bet high with probability m_H/b_1 , low with probability m_L/b_1 and fold with probability $1 - (m_H + m_L)/b_1$.

For Player II:

- (a) call a low bet if and only if $Y > b_1$.
- (b) call a high bet if and only if $Y > b_2$.

If this is the correct form of the strategies, the equations for these five numbers are easily found using the principle of indifference.

Indifference of II when $Y = b_1$ between fold and call low:

$$0 = m_L(B_1 + 2) - (b_3 - b_1)B_1. \tag{9}$$

Indifference of II when $Y = b_2$ between fold and call high:

$$0 = m_H(B_2 + 2) - (1 - b_3)B_2. \quad (10)$$

Indifference of I when $X = b_3$ between betting low and betting high:

$$2b_1 + (b_3 - b_1)(B_1 + 2) - (1 - b_3)B_1 = 2b_2 + (b_3 - b_2)(B_2 + 2) - (1 - b_3)B_2. \quad (11)$$

Indifference of I for $X \leq b_1$ between fold and bet low:

$$0 = 2b_1 - (1 - b_1)B_1. \quad (12)$$

Indifference of I for $X \leq b_1$ between fold and bet high:

$$0 = 2b_2 - (1 - b_2)B_2. \quad (13)$$

The solution of equations (9) through (13) is

$$\begin{aligned} b_1 &= B_1/(B_1 + 2) \\ b_2 &= B_2/(B_2 + 2) \\ b_3 &= 1 - 2/[(B_1 + 2)(B_2 + 2)] \\ m_H &= b_2(1 - b_3) \\ m_L &= b_1(b_3 - b_1) \end{aligned} \quad (14)$$

These are the actual optimal strategies provided the formula for the fold probability of Player I does not evaluate to a negative number, i.e. provided $m_H + m_L \leq b_1$. This condition may be written

$$B_2 - B_1 \leq B_1^2(B_2 + 2)^2/4. \quad (15)$$

Bellman and Blackwell assume $B_1 \geq c \approx .618\dots$, where c is the root of $c^3/4 - 2c^2 + 4c - 2 = 0$ in the interval $(0, 1)$, this being the smallest value of B_1 for which (15) holds for all $B_2 > B_1$. Karlin and Restrepo (1957) (see also Karlin (1959)) extend the Bellman-Blackwell analysis to an arbitrary number of bet sizes, but make the restriction that $B_1 \geq 1$.

Under condition (15), Player II's optimal strategy is unique, but Player I has many optimal strategies; he may do as he likes with $X < b_1$ provided the proportion of times he bets high is m_H/b_1 , the proportion of times he bets low is m_L/b_1 and the proportion of times he folds is $1 - (m_H + m_L)/b_1$. Admissible optimal strategies require folding with the worst hands. One admissible optimal strategy can be recommended from a practical point of view. If Player II makes the mistake of calling with a $Y < b_1$, it is much more likely that she will call a low bet. Therefore, to take advantage of this, Player I should bet low with his highest hands below b_1 . Thus, we suggest that Player I bet low if $b_1 - m_L < X < b_1$, bet high if $b_1 - m_L - m_H < X < b_1 - m_L$, and fold if $0 < X < b_1 - m_L - m_H$. This leads to the overall betting strategy:

If $0 < X < b_1 - m_L - m_H$, fold. If $b_1 - m_L - m_H < X < b_1 - m_L$, bet high.
 If $b_1 - m_L < x < b_3$, bet low. If $b_3 < X < 1$, bet high.

We now investigate what happens when (15) is not satisfied. When (15) is satisfied with equality, we have $m_H + m_L = b_1$, and the optimal strategy of Player I above does not permit him to fold. This remains true when (15) is not satisfied. The form of the optimal strategies of the players are as before, with only part (c) of Player I's optimal strategy modified. They depend on four numbers, $b_1 < b_2 < b_3$ and m_H .

For Player I:

- (a) bet high if $X > b_3$.
- (b) bet low if $b_1 < X < b_3$.
- (c) If $x < b_1$, bet high with probability m_H/b_1 and bet low with probability $1 - (m_H/b_1)$.

For Player II:

- (a) call a low bet if and only if $Y > b_1$.
- (b) call a high bet if and only if $Y > b_2$.

When (15) is not satisfied, the indifference equations (12) and (13) are replaced by the single equation:

Indifference of I for $X \leq b_1$ between betting low and betting high:

$$2b_1 - B_1(1 - b_1) = 2b_2 - B_2(1 - b_2). \quad (16)$$

Equations (9), (10), (11), and (16) with $m_L = b_1 - m_H$, are still linear in the unknowns but the solution is more complex:

$$\begin{aligned} m_H &= B_2(B_1 + 2)/D \\ b_1 &= m_H + (B_1(B_1 + 1)(B_2 + 2)/D) \\ b_2 &= b_1 + ((B_1 + 2)(B_2 - B_1)(B_2 + 2)/D) \\ b_3 &= 1 - ((B_1 + 1)(B_2 + 2)/D) \end{aligned}$$

where

$$D = B_1^2(B_2 + 2) + B_1(B_2^2 + 6B_2 + 6) + 2(B_2 + 1)(B_2 + 4).$$

When $B_1 = 0$, this reduces to the optimal strategies of Theorem 2.

Among the admissible optimal strategies of Player I, there is one that best takes advantage of Player II calling a low bet with $Y < b_1$. The recommended strategy is:

If $0 < X < m_H$, bet high. If $m_H < X < b_3$ bet low. If $b_3 < X < 1$, bet high.

5. General Distribution of Hands

In this section, we investigate the poker models of Borel and von Neumann when we drop the assumption that the hands are independent and identically distributed. La Relance with independent non-identically distributed hands has been treated by Karlin (1959, Exercise 9.3) and Sakaguchi (1984). The study of the dependent case of La Relance has been initiated by Sakaguchi and Sakai (1981), who treat only the special case in which the joint distribution of the hands has a Farlie-Gumbel-Morgenstern (FGM) distribution. We review the basic result in the independent case, and extend it to the general dependent case.

5.1. La Relance with independent not identically distributed hands. We assume that X and Y are independent and that $P(X = Y) = 0$. Thus we drop the assumption of identical distributions but keep the assumption that ties occur with probability zero. Let $F(x)$ denote the distribution function of X , and $G(y)$ denote the distribution function of Y . Without loss of generality, we assume that F and G are continuous. One

may instead assume without loss of generality that one of the distributions is the uniform distribution on $[0, 1]$, but then the other distribution may have to have some point masses.

We use the principle of indifference to find the optimal strategies. As in the i.i.d. case, Player II has an optimal strategy of the form for some number c : call if $Y > c$ and fold if $Y < c$. In the main case where Player I occasionally folds, Player II chooses c to make Player I indifferent between betting and folding with hands $X < c$. If I bets with such an X , he wins 2 if II has $Y < c$ and loses B if II has $Y > c$. His expected winnings are in this case, $2G(c) - B(1 - G(c))$. On the other hand, if I folds he wins nothing. He will be indifferent between betting and folding if $2G(c) - B(1 - G(c)) = 0$. This leads to the equation

$$G(c) = \frac{B}{B + 2}. \quad (17)$$

Such a c always exists since G is continuous. There may be an interval of such c but any such c may be used.

Player I may bet with $X > c$. To find the proportion of the time that I should bet with $X < c$, use the fact that II should be indifferent between calling and folding with $Y = c$. If Player I bets a proportion, π , of the time with $X < c$, then $P(X < c | \text{I bets}) = F(c)\pi / (F(c)\pi + (1 - F(c)))$. Suppose $Y = c$. If Player II calls, she wins $(B + 2)$ with probability $P(X < c | \text{I bets})$ and loses B with probability $P(X > c | \text{I bets})$. If she folds, she wins nothing. Equating to zero her expected return for calling yields the equation

$$\pi = \frac{B(1 - F(c))}{(B + 2)F(c)}. \quad (18)$$

Note that $\pi < 1$ if and only if $F(c) > B/(2B + 2)$. We distinguish two cases.

Case 1. $F(c) \leq B/(2B + 2)$. Here π of (18) is at least 1, so Player I always bets. If Player I always bets, a call by Player II with $Y = y$ gives her an expected return $(B + 2)F(y) - B(1 - F(y))$ and a fold gives her 0. Her best response is to call if and only if $F(y) > B/(2B + 2)$.

Case 2. $F(c) > B/(2B + 2)$. This is the main case, where Player I bets a proportion π of the hands below c . But as in Section 2, he takes maximum advantage of mistakes of

Player II by betting with the larger hands. So he finds b such that $F(c) - F(b) = \pi F(c)$ and bets with all $X > b$. The equation determining b is

$$F(b) = (1 - \pi)F(c) = F(c)(1 + G(c)) - G(c) \quad (19)$$

Note that he always folds if $F(c) = 1$.

In summary, assume X and Y are independent and continuous, and let c be any solution of (17). If $F(c) \leq B/(2B + 2)$, then it is optimal for I to bet for all X and for II to call if and only if $F(Y) > B/(B + 2)$. Otherwise, an optimal strategy for I is to bet if and only if $F(X) > F(c)(1 + G(c)) - G(c)$ and for II to call if and only if $Y > c$.

Example: Take $F(x) = x^2$ on $[0,1]$ where $\theta > 0$, and let $G(y) = y$ on $[0,1]$. (This is as if Player I gets two cards, Player II gets only one card, and the highest card wins.) If $B = 2$, we are in case 1 where I always bets, and II calls if and only if $Y^2 > 1/3$ ($Y > 0.577\dots$). If $B = 4$, we are in case 2 where I bets if and only if $X^2 > 2/27$ ($X > .272\dots$) and II calls if and only if $Y > 2/3$.

5.2. La Relance with negative dependence. Sakaguchi and Sakai (1981) treat this problem in the special case where the joint distribution of X and Y is an FGM family with density

$$f(x, y) = 1 + \theta(2x - 1)(2y - 1) \quad 0 < x < 1 \quad 0 < y < 1$$

where $-1 \leq \theta \leq 1$. The marginal distributions of X and Y are uniform. The parameter θ controls dependence. If $\theta = 0$, the variables are independent. The correlation between X and Y is $\theta/3$ so the maximum and minimum correlations that can be obtained are $1/3$ and $-1/3$.

Sakaguchi and Sakai found that when there is negative dependence ($\theta < 0$), there are optimal strategies of the following form for some ℓ and c with $\ell < c$.

$$\begin{aligned} \text{for I: Bet iff } X > \ell \\ \text{for II: Call iff } Y > c. \end{aligned} \quad (20)$$

We find below conditions on a general joint distribution of X and Y such that the optimal strategies have this same form. We assume the existence of a joint density.

We derive two equations that ℓ and c must satisfy if the above strategies are optimal. We get the first equation, in (22) below, using the indifference of Player I at ℓ . If $X = x$, Player I's payoff if he folds is zero. If $X = x \leq c$, the payoff if I bets is equal to

$$2P(Y < c|X = x) - BP(Y > c|X = x) = (B + 2)P(Y < c|X = x) - B.$$

If $X = x \geq c$, the payoff if I bets is equal to

$$\begin{aligned} 2P(Y < c|X = x) + (B + 2)P(c < Y < x|X = x) - BP(Y > x|X = x) \\ = -BP(Y < c|X = x) + 2(B + 1)P(Y < x|X = x) - B. \end{aligned}$$

Therefore, for I's strategy to be a best response to II's, we need

$$\begin{cases} (B + 2)P(Y < c|X = x) \leq B & \text{for } x \leq \ell \\ (B + 2)P(Y < c|X = x) \geq B & \text{for } \ell \leq x \leq c \\ BP(Y < c|X = x) + B \leq 2(B + 1)P(Y < x|X = x) & \text{for } x \geq c. \end{cases} \quad (21)$$

The case of $x = \ell$ gives one equation for ℓ and c :

$$P(Y < c|X = \ell) = B/(B + 2) \quad (22)$$

We get a second equation using indifference of Player II at c . Player II's strategy is obviously a best response to I's when $Y = y < \ell$. If Player II folds she wins nothing. If she calls with $Y = y > \ell$, she wins

$$(B + 2)P(\ell < X < y|Y = y) - BP(X > y|Y = y).$$

For II's strategy to be a best response to I's, we need therefore

$$\begin{cases} BP(X > y|Y = y) \geq (B + 2)P(\ell < X < y|Y = y) & \text{for } \ell \leq y \leq c \\ BP(X > y|Y = y) \leq (B + 2)P(\ell < X < y|Y = y) & \text{for } y \geq c \end{cases} \quad (23)$$

The case $y = c$ gives the second equation for ℓ and c :

$$(2B + 2)P(X > c|Y = c) = (B + 2)P(X > \ell|Y = c). \quad (24)$$

Suppose we can solve equations (22) and (24) simultaneously. This solution will then be optimal if (21) and (23) are satisfied. The first two inequalities in (21) will be satisfied if $P(Y < c|X = x)$ is increasing in x for $x < c$, a condition for negative association. This condition would be satisfied, for example, if the distribution of Y given $X = x$ is stochastically decreasing in x .

The inequalities in (23) will be satisfied if $P(X > y|Y = y, X > \ell)$ is decreasing in y for $y > \ell$. This condition is obviously satisfied if X and Y are independent, and generally satisfied for distributions with negative dependence. It is satisfied by the FGM family of distributions with negative association and the bivariate normal distributions with negative correlations.

The last inequality of (21) needs to be checked. From the second inequality of (21) evaluated at $x = c$, we find $B \leq (B + 2)P(Y < c|X = c)$. Hence from negative association, $B \leq (B + 2)P(Y < x|X = x)$ for $x > c$. But for $x > c$, we also obviously have $BP(Y < c|X = x) \leq BP(Y < x|X = x)$. Adding these two inequalities gives the last inequality of (21). We summarize this in the following theorem.

Theorem 3. *Suppose (22) and (24) are satisfied by some $\ell < c$. Suppose also that $P(Y < c|X = x)$ is nondecreasing in x , and $P(X > y|Y = y, X > \ell)$ is nonincreasing in y for $y > \ell$. Then the strategies given by (20) are optimal strategies for Players I and II. The value is*

$$V = P(X > Y) - P(Y > X) + BP(c < Y < X) - BP(\ell < X < Y, Y > c) \\ + 2P(X < Y < c, X > \ell) - 2P(Y < X < \ell).$$

Example: Suppose that the joint distribution of X and Y is bivariate normal with means zero, variances 1, and correlation coefficient $\rho = -0.6$, and suppose $B = 2$. Then the distribution of Y given $X = x$ is normal with mean ρx and variance $1 - \rho^2$, and the distribution of X given $Y = y$ is normal with mean ρy and variance $1 - \rho^2$. Equation (22) becomes $P(Y < c|X = \ell) = 1/2$, which shows that c is the median of the distribution of Y given $X = \ell$, namely $c = \rho\ell$. Substituting $\ell = c/\rho$ into equation (24) yields an equation for c , $6P(X > c|Y = c) = 4P(X > c/\rho|Y = c)$. Solving this for c with $\rho = -.6$

gives $c = .14632$ and $\ell = -.24387$. Thus Player I bets if $X > -.24387$ and II calls if $Y > .14632$. This may be compared with the independent case in which $\rho = 0$. Player I bets if $X > -.6745$ (the first quartile of $\text{normal}(0,1)$), and II calls if $Y > 0$.

5.3. The von Neumann model with non-identically distributed hands. We now consider the model with betting tree given by Figure 2, but we allow the variables X and Y to have different distributions. Let $F(x)$ denote the distribution function of X , the hand of Player I, and let $G(y)$ denote that of Y , the hand of Player II.

This is the von Neumann model if $F = G$ and $P(X = Y) = 0$. We drop the assumption that $F = G$ but keep the assumption that $P(X = Y) = 0$. One may then assume without loss of generality that both distributions are continuous. One may also assume that one of the distributions is the uniform distribution, but then the other distribution may have to have point masses. The conjectured optimal strategy for Player I has the form for some numbers a and b with $a \leq b$: if $a < X < b$ check, otherwise bet. The conjectured optimal strategy for Player II has the form for some number c with $a < c < b$: if Player I bets, then fold if $Y < c$ and call otherwise.

The indifference equations are

$$\text{at } a: (B + 2)G(c) = 2G(a) + B.$$

$$\text{at } b: 2G(b) = G(c) + 1.$$

$$\text{at } c: (B + 2)F(a) = B(1 - F(b)).$$

The top two equations simplify if G is the uniform distribution on $[0, 1]$. So take $G(y) = y$ for all $0 < y < 1$ and allow F to have point masses in $[0, 1]$. The top two equations become

$$\begin{aligned} c &= 2b - 1, \\ a &= (B + 2)b - (B + 1). \end{aligned} \tag{25}$$

The third equation becomes

$$F((B + 2)b - (B + 1)) = \frac{B}{B + 2}(1 - F(b)). \tag{26}$$

The left side is nondecreasing in b and the right side is nonincreasing in b , but since F may have discontinuities, one should define b as a root of (26) or as the point at which $F((B+2)x - (B+1)) - \frac{B}{B+2}(1 - F(x))$ changes sign.

There are two degenerate cases. One case is $F((B+1)/(B+2)) = 1$. Then (26) is satisfied with $b=(B+1)/(B+2)$, and $a = 0$. Since Player I never gets hands $x > (B+1)/(B+2)$, he never bets, i.e. he always checks, so Player II never gets to act. An optimal strategy for Player II is to call with any $Y > c = B/(B+2)$. This is sufficient to make sure that Player I never bets.

The other degenerate case is $F(1^-) \leq B/(2B+2)$, i.e. Player I has mass at least $(B+2)/(2B+2)$ at $x = 1$. In this case, $a = b = c = 1$ so that Player I always bets and Player II always folds.

In all other cases, that is if $F((B+1)/(B+2)) < 1$ and $F(1^-) > B/(2B+2)$, then b is determined in $(B+1)/(B+2) \leq b < 1$ by (26). However, to describe the optimal strategy of Player I, it is necessary to refine equation (26). The problem is that after solving (26) for b as the point at which the difference of the two sides changes sign, we may not have equality in (26) because there is a jump in F at a or b or both. We must have equality to have Player II indifferent at c . Therefore, we allow Player I to randomize by betting with probability π_1 with hands $X = a$ and betting with probability π_2 with hands $X = b$. We replace (26) by

$$P(X < a) + \pi_1 P(X = a) = \frac{B}{B+2} [P(X > b) + \pi_2 P(X = b)] \quad (26')$$

where $a = (B+2)b - (1+B)$. There always exists an appropriate choice of b and of the probabilities π_1 and π_2 to satisfy (26') with equality. Any such choice gives an optimal strategy for Player I in the general case.

As an example of this, consider the distribution F that has mass δ at $X = 1$ and mass $1 - \delta$ at $X = 0$. This game is equivalent to the game called "classical bluffing situation" by Friedman (1971) and "basic endgame in poker" by Cutler (1976). It is treated also by Ferguson (1968). If $\delta \geq B/(2B+2)$, we are in the second degenerate case. Otherwise, b is determined uniquely as $b = (B+1)/(B+2)$, $a = 0$ and $c = B/(B+2)$. Since there is a

jump in F at a , we may choose the probability π_1 so that (26) is satisfied. The equation becomes

$$\pi_1(1 - \delta) = \frac{B}{B + 2}\delta,$$

which gives $\pi_1 = (B/(B + 2))\delta/(1 - \delta)$. The optimal strategy of Player I is to bet with $X = 1$ and to bet (bluff) with probability δ if $X = 0$.

5.4. The von Neumann model with dependent hands. In the extension of the previous section to allow the hands to be dependent, the results are much weaker and the situation is more nebulous. Conditions under which the optimal strategies have the same form as in the same model with independent hands are difficult to interpret and check. We satisfy ourselves with a brief listing of these conditions under the assumption of the existence of the joint density of X and Y .

We assume that Player I bets if $X < a$ or $X > b$ and checks otherwise, while Player II calls if $Y > c$ and folds otherwise, where a , b and c are numbers for which $a < c < b$, and find conditions under which each strategy is a best response to the other.

The indifference equations are exactly analogous to those of the independent case, except that we must use conditional probabilities in them. They are

$$\text{at } a: (B + 2)P(Y < c | X = a) = 2P(Y < a | X = a) + B.$$

$$\text{at } b: 2P(Y < b | X = b) = P(Y < c | X = b) + 1.$$

$$\text{at } c: (B + 2)P(X < a | Y = c) = BP(X > b | Y = c).$$

If Player I bets with hand $X = x$, he expects to win

$$\begin{cases} (B + 2)P(Y < c | X = x) - B & \text{if } x \leq c \\ -BP(Y < c | X = x) + (2B + 2)P(Y < x | X = x) - B & \text{if } x > c. \end{cases}$$

If I checks, he expects to win $2P(Y < x | X = x)$ for all x . Therefore, I's strategy is a best response to II's strategy if

$$\text{For } x \leq a, \quad (B + 2)P(Y < c | X = x) - B \geq 2P(Y < x | X = x)$$

$$\text{For } a \leq x \leq c, \quad (B + 2)P(Y < c | X = x) - B \leq 2P(Y < x | X = x)$$

$$\text{For } c \leq x \leq b, \quad 2P(Y < x | X = x) \leq 1 + P(Y < c | X = x)$$

$$\text{For } b \leq x, \quad 2P(Y < x \mid X = x) \geq 1 + P(Y < c \mid X = x)$$

The first two indifference equations state that there is equality in these inequalities at $x = a$ and $x = b$. Therefore for the inequalities to be satisfied, it is sufficient that

$$BP(Y < c \mid X = x) + 2P(x < Y < c \mid X = x) \text{ be decreasing in } x \text{ for } x < c, \text{ and}$$

$$P(c < Y < x \mid X = x) + P(Y < x \mid X = x) \text{ be increasing in } x \text{ for } x > c.$$

These conditions cannot be written in terms of positive or negative association and so must be checked separately for each case.

If Player II calls with hand $Y = y$, she expects to win, (divided by $(P(X < a \mid Y = y) + P(X > b \mid Y = y))$ to make it conditional given player I bets)

$$\begin{aligned} (2B + 2)P(X < y \mid Y = y) - BP(X < a \mid Y = y) - BP(X > b \mid Y = y) & \quad \text{if } y < a \\ (B + 2)P(X < a \mid Y = y) - BP(X > b \mid Y = y) & \quad \text{if } a < y < b \\ (B + 2)[P(X < a \mid Y = y) + P(X > b \mid Y = y)] - (2B + 2)P(X > y \mid Y = y) & \quad \text{if } b < y. \end{aligned}$$

If II folds, she wins 0. Therefore, II's strategy is a best response to I's strategy if

$$\begin{aligned} (2B + 2)P(X < y \mid Y = y) & \leq BP(X < a \mid Y = y) + BP(X > y \mid Y = y) & \quad \text{if } y \leq a \\ (B + 2)P(X < a \mid Y = y) & \leq BP(X > b \mid Y = y) & \quad \text{if } a \leq y \leq c \\ (B + 2)P(X < a \mid Y = y) & \geq BP(X > b \mid Y = y) & \quad \text{if } c \leq y \leq b \\ (B + 2)[P(X < a \mid Y = y) + P(X > b \mid Y = y)] & \geq (2B + 2)P(X > y \mid Y = y) & \quad \text{if } b \leq y, \end{aligned}$$

The first and fourth inequalities require special checking. The middle two inequalities are satisfied with equality at $y = c$, and so they will be satisfied if $(B + 2)P(X < a \mid Y = y) - BP(X > b \mid Y = y)$ is increasing in y for $a < y < b$. This is satisfied for distributions with negative dependence, but *not* satisfied in the positive dependent case.

6. References.

Richard Bellman (1952) On games involving bluffing, *Rendiconti del Circolo Math. di Palermo* Ser. 2, Vol. 1 139-156.

- R. Bellman and D. Blackwell (1949) Some two-person games involving bluffing, *Proc. Nat. Acad. Sci.* **35**, 600-605. 8/04/49.
- Émile Borel (1938) *Traité du Calcul des Probabilités et ses Applications* Volume IV, Fascicule 2, Applications aux jeux des hazard, Gautier-Villars, Paris.
- William H. Cutler (1976) End-Game Poker, Preprint.
- T. S. Ferguson (1968) *Mathematical Statistics, A Decision Theoretic Approach*, Academic Press, New York. (Exercise 1.7.4.)
- L. Friedman (1971) Optimal bluffing strategies in poker, *Man. Sci.* **17**, B764-B771.
- Samuel Karlin (1959) *Mathematical Methods and Theory in Games, Programming and Economics*, in two vols., Reprinted 1992, Dover Publications Inc., New York.
- S. Karlin and R. Restrepo (1957) Multistage poker models, *Contrib. Theor. Games III* 337-363.
- Donald J. Newman (1959) A model for “real” poker, *Operations Research* **7**, 557-560.
- J. P. Ponsard (1975) Zero-sum games with almost perfect information, *Man. Sci.* **21**, 794-805.
- M. Sakaguchi (1984) A note on the disadvantage for the sente in poker, *Math. Japonica* **29**, 483-489.
- M. Sakaguchi and S. Sakai (1981) Partial information in a simplified two person poker, *Math. Japonica* **26**, 695-705.
- S. Sorin and J. P. Ponsard (1980) The LP formulation of finite zero-sum games with incomplete information, *Int. J. Game Theory* **9**, 99-105.
- J. von Neumann and O. Morgenstern (1944) *The Theory of Games and Economic Behavior*, Princeton University Press.