Large Sample Theory
Ferguson

Exercises, Section 18, Asymptotic Normality of the Maximum Likelihood Estimate.

1. (a) Let $X_1, \ldots, X_n$ be a sample from the negative binomial distribution with density $f(x|\theta) = \binom{x+r-1}{r-1}\theta^r(1-\theta)^x$ for $x = 0, 1, \ldots$, where r is known. What is the maximum likelihood estimate of θ? Find Fisher information. Find the asymptotic distribution of the maximum likelihood estimate.

(b) Let $Y_1, \ldots, Y_n$ be a sample from the binomial distribution with density $g(y|\theta) = \binom{k}{y}\theta^y(1-\theta)^{k-y}$ for $y = 0, 1, \ldots, k$ where k is known. What is the maximum likelihood estimate of θ and what is its asymptotic distribution?

(c) For what values of θ is the negative binomial sampling of part (a) asymptotically better than binomial sampling for use in estimating θ? (Suppose r and k are given numbers.)

2. Let $X_1, \ldots, X_n$ be a sample from a distribution with density

$$f(x|\theta_1, \theta_2) = \begin{cases} \frac{\theta_1}{\theta_2} e^{-x/\theta_2} & \text{for } x > 0 \\ \frac{(1-\theta_1)}{\theta_2} e^{x/\theta_2} & \text{for } x < 0 \end{cases}$$

where $0 < \theta_1 < 1$ and $\theta_2 > 0$.

(a) Show that (S, K) is sufficient for (θ_1, θ_2), where K is the number of positive X_i's, $K = \sum_1^n \mathrm{I}(X_i > 0)$, and $S = \sum_1^n |X_i|$.

(b) Find the maximum likelihood estimate, $(\hat{\theta}_1, \hat{\theta}_2)$, of (θ_1, θ_2).

(c) Find the Fisher information matrix, $\mathcal{I}(\theta_1, \theta_2)$.

(d) Find the asymptotic joint distribution of $(\hat{\theta}_1, \hat{\theta}_2)$.

3. The observations are $X_i = Z_i + \epsilon_i$, $i = 1, 2, \ldots, n$, where the Z_i are unobservable i.i.d. exponential random variables with mean $\theta > 0$ ($f_Z(z) = (1/\theta)\exp\{-z/\theta\}\mathrm{I}\{z > 0\}$), and the error terms ϵ_i are i.i.d. Bernoulli with parameter p, independent of the Z_i ($p = P(\epsilon_i = 1) = 1 - P(\epsilon_i = 0)$).

(a) Find the method of moments estimates of θ and p. For what values of (θ, p) are these estimates consistent?

(b) Show there is a two-dimensional sufficient statistic for (θ, p).

(c) Find Fisher Information.

4. Each of n light bulbs with common exponential density for the time to failure is left on until it fails or until time T whichever occurs first. Thus the observations, $X_1, \ldots, X_n$, (the 'on' times) are i.i.d. with density $(1/\theta)e^{-x/\theta}$ on $(0, T)$, and with $\mathrm{P}(X_i = T) = e^{-T/\theta}$.

(a) Find the MLE of θ based on $X_1, \ldots, X_n$.

(b) Find the asymptotic distribution of the MLE.

5. Let (X_i, Y_i), $i = 1, \ldots, n$, be a sample from a distribution with density $f(x, y|\mu, \theta)$. Suppose the marginal distribution of X depends only on μ, and the the conditional distribution of Y given X depends only on θ, so that the density can be written in the form $f(x, y|\mu, \theta) = g(x|\mu)h(y|x, \theta)$.

(a) Show that the Fisher information matrix is diagonal, so that the maximum likelihood estimates of μ and θ are asymptotically independent.

(b) Consider the model $Y = \beta X + e$, where $X \in \mathcal{N}(\mu, \sigma_x^2)$, $e \in \mathcal{N}(0, \sigma_e^2)$ and X and e are independent. The parameters $\mu, \sigma_x^2, \beta, \sigma_e^2$ are all unknown. Find Fisher Information.

6. (a) Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample from a bivariate distribution with mean (μ_x, μ_y), and covariance matrix, $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}$. As an estimate of $\theta = \mu_x/\mu_y$, one may use simply $\theta_n^* = \overline{X}_n/\overline{Y}_n$. Assume $\mu_y \neq 0$, and find the asymptotic distribution of θ_n^*.

(b) Specialize to the case where the distribution of Y_i is exponential with mean 1, and where the conditional distribution of X_i given $Y_i = y_i$ is normal with mean, θy_i, and variance, 1. First note that $\mathrm{E}(X_i) = \theta$ and $\mathrm{E}(Y_i) = 1$, so that $\mu_x/\mu_y = \theta$. Find Σ and the asymptotic distribution of θ_n^* in this case.

(c) In case (b), find the maximum likelihood estimate of θ and compare its asymptotic distribution to the asymptotic distribution of θ_n^* found in (b).