Large Sample Theory
Ferguson

Exercises, Section 17, Strong Consistency of Maximum Likelihood Estimates.

1. (Which double exponential distribution is closest to the standard normal?) Find the Kullback-Leibler information number, $K(f_0, f_1)$, where f_0 is the density of the standard normal distribution and f_1 is the density of the double exponential distribution with density $f_1(x|\theta) = (2\theta)^{-1}e^{-|x|/\theta}$. What value of θ minimizes $K(f_0, f_1)$? This is the value of θ for which $f_1(x|\theta)$ is hardest to distinguish from the standard normal distribution asymptotically when the standard normal distribution is the true distribution.

2. Let $X_1, X_2, \ldots$ be a sample from a Cauchy distribution with median θ. Use Theorem 17 to show that the maximum likelihood estimate of θ is strongly consistent. (One possibility is to compactify the parameter space by adding points at $-\infty$ and $+\infty$, and choosing the distribution of X given θ to be degenerate at θ when $\theta = \pm\infty$.)

3. Let $f(x|\theta)$ be the density of a location parameter family of distributions on the real line, $f(x|\theta) = f(x - \theta)$. Assume (1) $f(x)$ is upper semi-continuous in x, (2) $f(x) \to 0$ as $x \to \pm\infty$ and (3) $\int_{-\infty}^{\infty}(\log(f(x))f(x)\,dx > -\infty$. Let $\hat{\theta}_n$ denote a maximum likelihood estimate of θ based on a sample of size n from the distribution. Show that $\hat{\theta}_n$ is a stongly consistent estimate of θ.

4. **Bayesian Testing.** For an unknown density $f(x)$, consider testing the simple hypothesis, $H_0 : f(x) = f_0(x)$, versus the simple hypothesis, $H_1 : f(x) = f_1(x)$, from a Bayesian point of view, where f_0 and f_1 are given distinct densities. Let p_0 be the prior probability that H_1 is true and $1 - p_0$ be the prior probability that H_0 is true.

(a) Find p_n the posterior probability that H_1 is true, given a sample, $X_1, X_2, \ldots, X_n$, from $f(x)$.

(b) Show that if H_0 is true, $p_n \xrightarrow{a.s.} 0$ exponentially fast at rate $K(f_0, f_1)$ (i.e. show $-\frac{1}{n}\log p_n \xrightarrow{a.s.} K(f_0, f_1)$), where K is Kullback-Leibler information.