Large Sample Theory
Ferguson

Exercises, Section 14, Asymptotic Theory of Extreme Order Statistics.

1. Let $X_1, \ldots, X_n$ be a sample from a distribution with distribution function $F(x)$ and let $M_n = \max\{X_1, \ldots, X_n\}$ be the maximum of the sample. Find a normalization, $(M_n - a_n)/b_n$, if any exists, that has a nondegenerate limiting distribution as $n \to \infty$, for the following distributions:
(a) The logistic distribution with density $f(x) = e^{-x}/(1 + e^{-x})^2$.
(b) The distribution with distribution function $F(x) = 1 - (\log(x + 1))/x$ for $x > 0$.
(c) The cosine distribution with density $f(x) = (1/2)\cos(x)\,\mathrm{I}(-\pi/2 < x < \pi/2)$.

2. Suppose X has the $G_{1,\gamma}(x)$ distribution, and let $Y = \gamma(X - 1)$. Show that as $\gamma \to \infty$, Y converges in law to the G_3 distribution.

3. Suppose in Example 6 that $F(x) = \Phi(x-\mu)$ so that we are sampling from a normal distribution with mean μ.
(a) Find the asymptotic distribution of M_n.
(b) Show that

$$\hat{\mu}_n \stackrel{\text{def}}{=} M_n - \sqrt{2\log n} \xrightarrow{P} \mu.$$

Thus $\hat{\mu}_n$ is a consistent estimate of μ. What is its asymptotic efficiency relative to $\overline{X}_n$?

4. Let $X_1, \ldots, X_n$ be a sample from the distribution on the interval $(-\pi/2, 0)$, with distribution function $F(x) = \cos(x)$ for $-\pi/2 < x < 0$, and let M_n represent the maximum of the sample.
(a) What is the distribution function of M_n?
(b) Find b_n such that M_n/b_n converges in law to a nondegenerate distribution and find the distribution.

5. Let $X_1, X_2, \ldots$ be i.i.d. from a distribution with density $f(x)$ such that $f(x) = 0$ for $x < a$, $f(a) > 0$ and $f(x)$ is right continuous at a. Show that $n[\min\{X_1, \ldots, X_n\} - a]$ converges in law to the exponential distribution with rate parameter $f(a)$ (i.e. mean $1/f(a)$).

6. What can you say about the asymptotic distribution of $\min\{X_1, \ldots, X_n\}$ when $X_1, \ldots, X_n$ are i.i.d. with distribution G_3? What is the exact distribution?