Large Sample Theory
Ferguson

Exercises, Section 5, Central Limit Theorems.

1. (a) Using a Chebyshev's-Inequality-like argument, show that (assuming the expectations exist) $\mathrm{E}|X|^{2+\alpha} \geq t^\alpha \mathrm{E}[X^2 \mathrm{I}(|X| \geq t)]$ for all $\alpha > 0$ and $t > 0$.

(b) Using part (a) and Lindeberg, prove Liapounov's Theorem: Let $X_{n1}, X_{n2}, \ldots, X_{nn}$ be independent with $\mathrm{E}X_{nj} = 0$ and $\mathrm{E}|X_{nj}|^{2+\alpha} < \infty$ for some $\alpha > 0$ and all n and j. Let $Z_n = \sum_{j=1}^n X_{nj}$ and $B_n^2 = \mathrm{Var} Z_n = \sum_{j=1}^n \mathrm{Var} X_{nj}$. Then $Z_n/B_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$, provided $\dfrac{1}{B_n^{2+\alpha}} \displaystyle\sum_{j=1}^n \mathrm{E}|X_{nj}|^{2+\alpha} \to 0$ as $n \to \infty$.

2. Let $X_1, X_2, \ldots$ be independent exponential random variables with means $\beta_1, \beta_2, \ldots$ respectively, and let $Z_n = X_1 + \cdots + X_n$. Show that if $\max_{1 \leq j \leq n} \beta_j^2 / \sum_{j=1}^n \beta_j^2 \to 0$ as $n \to \infty$, then $(Z_n - \mathrm{E}Z_n)/\sqrt{\mathrm{Var} Z_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. (Use Liapounov's Theorem with $\alpha = 2$.)

3. (a) Let $X_1, X_2, \ldots$ be independent Poisson random variables with means $\lambda_1, \lambda_2, \ldots$ respectively, and let $Z_n = X_1 + \cdots + X_n$. Show that $(Z_n - \mathrm{E}Z_n)/\sqrt{\mathrm{Var} Z_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ if and only if $\sum_1^n \lambda_j \to \infty$.

(b) Show that this can provide an example to show you can get asymptotic normality without the Lindeberg condition being satisfied.

4. As an illustration of the use of Kendall's tau, here is a famous little example taken from M. G. Kendall's 1948 book, *Rank Correlation Methods*. Suppose a number of boys are ranked according to their ability in mathematics and music. Such a pair of rankings for ten boys, denoted by the letters A to J, might be as follows :

Boy :	A	B	C	D	E	F	G	H	I	J
Maths. :	7	4	3	10	6	2	9	8	1	5
Music :	5	7	3	10	1	9	6	2	8	4

Compute T_n, the number of discrepencies, and τ_n, Kendall's rank correlation coefficient. According to Kendall's tables, $P(T_n \geq 32) = .054$ for $n = 10$. Compare this probabilty with the normal approximation, using the correction for continuity.

5. Let X be a Poisson random variable with mean $\lambda = 10$.

(a) Find the exact probability, $P(X \leq 10)$. (You may use the calculators found on the web page http://www.math.ucla.edu/ tom/distributions/CONTENTS.html)

(b) Find the normal approximation to $P(X \leq 10)$.

(c) Find the first Edgeworth approximation to $P(X \leq 10)$.

(d) Find the second Edgeworth approximation to $P(X \leq 10)$. (Please make the corrections for continuity in all these approximations.)

6. (a) Let $X_1, X_2, \ldots$ be i.i.d. with $\mathrm{E}X_i = 0$ and $\mathrm{Var} X_i = 1$. Let $S_n = \sum_{j=1}^n a_{nj} X_j$ and $T_n = \sum_{j=1}^n b_{nj} X_j$, where a_{nj} and b_{nj} are constants, normalized so that $\sum_{j=1}^n a_{nj}^2 =$

$\sum_{j=1}^n b_{nj}^2 = 1$. Let $\rho_n = \sum_{j=1}^n a_{nj}b_{nj}$. Assume that $\rho_n \to \rho$, $\max_{j\le n} a_{nj}^2 \to 0$ and $\max_{j\le n} b_{nj}^2 \to 0$ as $n \to \infty$. Show that

$$(S_n, T_n) \xrightarrow{\mathcal{L}} \mathcal{N}((0,0), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}).$$

(b) Apply the above to find the asymptotic joint distribution of $\sum_1^n X_j$ and $\sum_1^n jX_j$.

7. Let $X_1, X_2, \ldots$ be independent random variables with X_n having a uniform distribution over the interval $[-n, n]$.

(a) Does $\overline{X}_n \xrightarrow{\text{P}} 0$ as $n \to \infty$?

(b) Does $\sqrt{n}\overline{X}_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$ for some number σ^2? If not, what can you say about the large sample distribution of $\overline{X}_n$? (Maybe you should answer (a) after (b).)

8. *The Coupon Collector's Problem.* Coupons are drawn at random with replacement from among N distinct coupons until exactly n distinct coupons are observed. Let S_n denote the total number of coupons drawn. Then $S_n = Y_1 + \cdots + Y_n$, where Y_j is the number of coupons drawn after observing $j-1$ distinct coupons until the jth distinct coupon is drawn. Then $Y_1, \ldots, Y_n$ are independent geometric random variables with means, $\text{E}Y_j = N/(N-j+1)$, and variances, $\text{Var}(Y_j) = N(j-1)/(N-j+1)^2$. Let $n = \lceil Nr \rceil$ for some fixed $r \in (0,1)$, and let N, and hence n, tend to ∞. Show $\sqrt{n}((S_n/n) - m) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$ and find m and σ^2 as functions of r.

9. (a) Assume for a triangular array of independent variables that the X_{nj} are uniformly bounded, say $|X_{n,j}| < A$ for all n and j and some fixed constant A. Let $S_n = \sum_{j=1}^n X_{nj}$. Show that

$$\frac{S_n - \text{E}(S_n)}{\sqrt{\text{Var}(S_n)}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \qquad \text{provided} \qquad \text{Var}(S_n) \to \infty.$$

(b) Apply this to the binomial random variable, $Y_n \in \mathcal{B}(n, p_n)$, (which is a sum of independent Bernoullis) in the case $p_n = 1/\sqrt{n}$ to show that

$$\sqrt[4]{n}\left(\frac{Y_n}{\sqrt{n}} - 1\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

10. Let $X_1, X_2, \ldots, X_n$ be a sample from a distribution with distribution function $F(x)$ and density $f(x)$. A simple estimate of the density at a point x is given by $\hat{f}_n(x) = \dfrac{\hat{F}_n(x+b_n) - \hat{F}_n(x-b_n)}{2b_n}$, where $\hat{F}_n(x)$ is the sample distribution function. Here, b_n is a sequence of constants tending to zero at an appropriate rate. Note that $Z_n = n(\hat{F}_n(x+b_n) - \hat{F}_n(x-b_n))$ has a binomial distribution, $\mathcal{B}(n, p_n)$ where $p_n = F(x+b_n) - F(x-b_n)$. Assume that x is a point of continuity of f and that $f(x) > 0$.

(a) Using the preceding exercise, show that $\sqrt{2nb_n}(\hat{f}_n(x) - \text{E}\hat{f}_n(x)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, f(x))$, provided $b_n \to 0$ and $nb_n \to \infty$.

(b) Assuming that $f(x)$ is differentiable a suitable number of times, find extra conditions on b_n such that is it true that $\sqrt{2nb_n}(\hat{f}_n(x) - f(x)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, f(x))$.

11. Let $X_1, X_2, \ldots$ be independent with $\mathrm{P}(X_n = n) = \mathrm{P}(X_n = -n) = p_n/2$ and $\mathrm{P}(X_n = 0) = 1 - p_n$, and let $Z_n = X_1 + \cdots + X_n$. Take $p_n = 1/n^2$.

(a) Show, using the converse to the Lindeberg-Feller Theorem, that Z_n/B_n is not asymptotically normal.

(b) What can you say about the asymptotic distribution of Z_n or Z_n/B_n?

12. Show that the Lindeberg condition implies the uniformly asymptotically negligiblity (UAN) condition: $\max_{j \le n} \sigma_{nj}^2/B_n^2 \to 0$ as $n \to \infty$.