Large Sample Theory
Ferguson

Exercises, Section 1, Modes of Convergence.

1. (a) The case $r = 1$ of Chebyshev's Inequality is known as Markov's Inequality and is usually written $\mathrm{P}(|X| \geq \epsilon) \leq \mathrm{E}(|X|)/\epsilon$ for an arbitrary random variable X and $\epsilon > 0$. For each fixed $\epsilon > 1$, find a distribution for X with $\mathrm{E}X = 0$ and $\mathrm{E}|X| = 1$ that gives equality in Markov's inequality.

(b) Prove for an arbitrary random variable X,

$$\mathrm{P}(|X| \geq \epsilon) \leq \frac{\mathrm{E}\cosh(X) - 1}{\cosh(\epsilon) - 1}.$$

2. Suppose that $f_n(x)$ and $g(x)$ are densities, and that for all x, $f_n(x)$ converges to some constant (independent of x) times $g(x)$. Does it follow that the random variable with density $f_n(x)$ converges in law to the random variable with density $g(x)$? If so, show it. If not, give a counterexample.

3. Suppose that X_n has the binomial distribution, $\mathcal{B}(n, p)$ for some $0 < p < 1$.

(a) For fixed k, find $\lim_{n\to\infty} \mathrm{P}(X_n \leq k - 1 | X_n \leq k)$.

(b) Let the distribution of Y_n be the conditional distribution of X_n given $X_n \leq k$. Express the result in (a) in the form $Y_n \xrightarrow{\mathcal{L}} Y$.

4. Let X have an inverse power distribution with distribution function, $F_X(x) = 1 - x^{-\alpha}$ for $x \in [1, \infty)$, where $\alpha > 0$.

(a) Show $\mathrm{E}X = \alpha/(\alpha - 1)$ for $\alpha > 1$, and $\mathrm{Var}(X) = \alpha/((\alpha - 2)(\alpha - 1)^2)$ for $\alpha > 2$.

(b) Let $Y = (\alpha - 1)X - \alpha$ (so that Y has mean 0 and variance tending to 1 as $\alpha \to \infty$. Show that $Y \xrightarrow{\mathcal{L}} Z$ as $\alpha \to \infty$ for some random variable Z and find the distribution function of Z.

5. We say $X_n \xrightarrow{P} \infty$ if for every number B (no matter how big), $\mathrm{P}(X_n > B) \to 1$ as $n \to \infty$. We say $X_n \xrightarrow{a.s.} \infty$ if for every number B, $\mathrm{P}(X_k > B$ for every $k \geq n) \to 1$ as $n \to \infty$.

Suppose $X_1, X_2, \ldots$ are independent with $\mathrm{P}(0 \leq X_j \leq 1) = 1$ for all j. Let $S_n = \sum_1^n X_j$ and $\mu_n = \mathrm{E}S_n$. Show that if $\mu_n \to \infty$ as $n \to \infty$, then $S_n \xrightarrow{\mathrm{P}} \infty$. (Hint: Show $\mathrm{Var}(X_i) \leq \mathrm{E}X_i$ and use Chebyshev's inequality.) Show that this implies $S_n \xrightarrow{a.s.} \infty$.