NEGATIVE DEPENDENCE, and the GEOMETRY of POLYNOMIALS

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J. Borcea, P. Brändén and T. M. Liggett, *Negative dependence and the geometry of polynomials.*

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R. Pemantle, *Towards a theory of negative dependence*, J. Math. Phys. **41** (2000), 1371–1390.

D. Wagner, *Negatively correlated random variables and Mason's conjecture for independent sets in matroids*, Ann. Combin.

Mason's 1972 Conjecture: From Wagner's paper: The sequence

$$(I_k : 0 \leq k \leq r)$$

of numbers of independent k-sets of an m-element rank r matroid is ultra log concave, in the sense that

$$\frac{I_k^2}{\binom{m}{k}^2} \geq \frac{I_{k-1}}{\binom{m}{k-1}^2} \frac{I_{k+1}}{\binom{m}{k+1}^2}.$$

A Primer on Positive Dependence

Definitions: For a probability measure μ on $\{0,1\}^n$,

(a) (Positive) lattice condition (PLC):

$$\mu(\eta \wedge \zeta)\mu(\eta \vee \zeta) \geq \mu(\eta)\mu(\zeta).$$

(b) Association: For all $\uparrow F, G$,

$$\int FG d\mu \geq \int F d\mu \int G d\mu.$$

FKG Theorem. *PLC $\Rightarrow$ association.*

Harris' Theorem. *For any attractive spin system on $\{0,1\}^n$, if the initial distribution is associated, then so is the distribution at later times.*

Corollary. *The upper invariant measure of the contact process is associated.*

On to Negative Dependence

Definitions: For a probability measure μ on $\{0,1\}^n$,

(a) Negative lattice condition (NLC):

$$\mu(\eta \wedge \zeta)\mu(\eta \vee \zeta) \leq \mu(\eta)\mu(\zeta).$$

(b) Negative association (NA): For all $\uparrow F, G$ **depending on disjoint sets of coordinates**

$$\int FG d\mu \leq \int F d\mu \int G d\mu.$$

Main problems:

(a) NLC does **not** imply NA.

(b) The symmetric exclusion process does **not** preserve NA.

Previously known negative correlation result for the symmetric exclusion process. Andjel (1988) proved for $A \cap B = \emptyset$,

$$P^\eta(\eta_t \equiv 1 \text{ on } A \cup B) \leq P^\eta(\eta_t \equiv 1 \text{ on } A)P^\eta(\eta_t \equiv 1 \text{ on } B).$$

The same approach does **not** give

$$P^\eta(\eta_t \equiv 1 \text{ on } A, \eta_t \equiv 0 \text{ on } B) \geq$$
$$P^\eta(\eta_t \equiv 1 \text{ on } A)P^\eta(\eta_t \equiv 0 \text{ on } B).$$

Connection to Polynomials

The **generating polynomial** of μ is

$$f(z_1, ..., z_n) = f(z) = E^{\mu} z^{\eta} = E^{\mu} \prod_{j=1}^{n} z_j^{\eta(j)}.$$

Definition. μ is (a) strong Rayleigh, (b) Rayleigh, or (c) weak Rayleigh if for $j \neq k$,

$$(*) \qquad \frac{\partial f}{\partial z_j}(z) \frac{\partial f}{\partial z_k}(z) \geq f(z) \frac{\partial^2 f}{\partial z_j \partial z_k}(z)$$

for all (a) $z_j \in (-\infty, \infty)$, (b) $z_j \geq 0$, or (c) $z_j = 0, 1, \infty$.
Note: If $z_j \equiv 1$, then (*) says $E^{\mu}\eta(j)\eta(k) \leq E^{\mu}\eta(j)E^{\mu}\eta(k)$.

Glossary

Pemantle (2000) used the following terminology and notation:

1. h-NLC: hereditary NLC, i.e., all projections are NLC.
2. h-NLC+: h-NLC after application of external fields.
3. CNA: NA after conditioning on some coordinates.
4. CNA+: CNA after application of external fields.

Note: Weak Rayleigh = h-NLC, and Rayleigh = h-NLC+. There is no probabilistic interpretation of strong Rayleigh.

5. ULC: $\mu\{\eta : \sum_j \eta(j) = k\}/\binom{n}{k}$ is logconcave. This is equivalent to symmetrization of μ is NLC.

Some results from Pemantle (2000):

$$\begin{array}{ccc} \text{CNA+} & \Longrightarrow & \text{Rayleigh } (= h\text{-NLC+}) \\ \Downarrow & & \Downarrow \\ \text{CNA} & \Longrightarrow & \text{weak Rayleigh } (=h\text{-NLC}) \\ \Downarrow & & \Downarrow \\ \text{NA} & & \text{NLC} \end{array}$$

and the top four properties and ULC are equivalent for exchangeable μ.

Pemantle's conjectures: (a) The horizontal implications are equivalences. (b) Any of the top four properties implies ULC.

Wagner's "big" conjecture: Rayleigh $\Longrightarrow$ ULC. (This would imply a conjecture due to Mason (1972) for a large class of matroids.)

Some new results:

(a) None of the above six properties implies ULC.

(b) Strong Rayleigh implies NA and ULC.

(c) If the initial distribution of a symmetric exclusion process is strong Rayleigh, then so is the distribution at later times.

(d) Statement (c) is false for the Rayleigh property.

(e) The horizontal implications are equivalences for "almost" exchangeable μ.

Remarks on the Counterexample(s)

1. Our original example is on $n = 20$ sites, and is a bit hard to describe.

2. Later, Kahn and Neiman gave simpler examples with $n = 2k, \beta \in (0,1)$:

$$\mu(\eta) \sim \begin{cases} 1 & \text{if } |\eta| = k-1, \eta(1) = 1 \text{ or } |\eta| = k+1, \eta(1) = 0, \\ \beta & \text{if } |\eta| = k, \\ \beta^2 & \text{if } |\eta| = k-1, \eta(1) = 0 \text{ or } |\eta| = k+1, \eta(1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(a) μ is CNA+ iff $\beta \geq \frac{1}{\sqrt{2}}$,

and

(b) μ is ULC iff $\beta \geq 1 - \frac{2}{k+1}$.

This gives a counterexample for $n = 12$.

3. Both examples are almost exchangeable.

Examples

(a) Lyons (2003) defined μ to be **determinantal** if there is a matrix M so that

$$\mu\{\eta \equiv 1 \text{ on } A\} = \det(\text{submatrix of } M \text{ determined by } A).$$

He proved that if M is a positive contraction, then μ is CNA+, and hence Rayleigh. In fact, it is strong Rayleigh.

(b) The uniform spanning tree measure is strong Rayleigh.

(c) The random cluster measure on a graph $G = (V, E)$ has

$$\mu(\eta) \sim \left(\prod_{j\in E} p_j^{\eta(j)}(1-p_j)^{1-\eta(j)}\right) q^{C(\eta)},$$

where $C(\eta)$ = the number of components in the subgraph determined by η.

(i) If $q \geq 1$, PLC is satisfied, and therefore μ is associated.

(ii) If $q < 1$, NLC is satisfied but other properties are conjectural. If $G = K_3$,

$$\frac{\partial f}{\partial z_1}(z)\frac{\partial f}{\partial z_2}(z) - f(z)\frac{\partial^2 f}{\partial z_1 \partial z_2}(z) = q^2(1-q)z_3(q+z_3),$$

so μ is Rayleigh but not strong Rayleigh. The model is Rayleigh for graphs with five or fewer vertices.

Why is strong Rayleigh better than Rayleigh?

Theorem (Brändén). *The probability measure μ is strong Rayleigh iff $f(z_1, ..., z_n) \neq 0$ for all (complex) $z_1, ..., z_n$ with strictly positive imaginary part. (I.e., f is stable.)*

Hint of proof. Write $z_k = x_k + iy_k$. Solve

$$f(z) = 0 \quad \text{and} \quad y_3 = 0, ..., y_n = 0$$

for y_1, y_2. Result:

$$y_1 f_1(x) = -y_2 f_2(x) \quad \text{and} \quad y_1 y_2 f_{1,2}(x) = f(x).$$

(a) $y_1, y_2 > 0 \Rightarrow f_1(x) f_2(x) < 0, f(x) f_{1,2}(x) > 0 \Rightarrow$ not strong Rayleigh.

(b) Not strong Rayleigh $\Rightarrow f_1(x) f_2(x) - f(x) f_{1,2}(x) < 0$ for some $x_3, ..., x_n$ (it does not depend on x_1, x_2). Choose x_1, x_2 so that $f_1(x) f_2(x) < 0, f(x) f_{1,2}(x) > 0$. Then $y_1 y_2 > 0$, and can take $y_1, y_2 > 0$.

Theorem. *Strong Rayleigh $\Rightarrow$ NA.*

Main elements of proof. (a) Symmetric homogenization.

(b) Feder-Mihail proof of NA in the context of "balanced matroids" – e.g., uniform spanning tree measure.

Theorem. *Strong Rayleigh $\Rightarrow$ ULC.*

Proof. $h(z) = f(z, z, ..., z) = E^\mu z^N$, where $N = \sum_k \eta(k)$, has only real zeros. This implies ULC by the Newton inequalities.

Theorem. *If μ is strong Rayleigh, then so is $\theta\mu + (1-\theta)\tau\mu$, where $\tau\mu$ is obtained from μ by permuting two coordinates (say 1,2).*

Hint of proof. Let

$$g(z) = \theta f(z) + (1-\theta) f(\tau z)$$

be the generating polynomial of the new measure. Need to show that $g(z) \neq 0$ if z_k has positive imaginary part for each k. Fix $z_3, ..., z_n$ in the upper half plane. Look at the transformation T_θ given by

$$T_\theta h(z_1, z_2) = \theta h(z_1, z_2) + (1-\theta) h(z_2, z_1).$$

Need: h (complex) stable implies $T_\theta h$ (complex) stable.

Now use: Suppose $h(z, w) = a + bz + cw + dzw$, where a, b, c, d are complex, and not all zero. Then h is stable if and only if

$$\Re(b\overline{c} - a\overline{d}) \geq |bc - ad|,\ \Im(a\overline{b}) \geq 0,\ \Im(a\overline{c}) \geq 0,\ \Im(b\overline{d}) \geq 0,\ \Im(c\overline{d}) \geq 0.$$

The Symmetric Exclusion Process on S

Theorem. *If μ is strong Rayleigh, then $\mu T(t)$, the distribution at time t is also strong Rayleigh.*

Proof. This follows from the previous result if the transition rate is zero except for one pair of sites. In general, use the Trotter product formula: If $T_k(t)$ has generator $\mathcal{L}_k$, then the semigroup with generator $\mathcal{L}_1 + \mathcal{L}_2$ is given by

$$T(t) = \lim_{n \to \infty} \Big[T_1(t/n) T_2(t/n) \Big]^n.$$

Application to stationary distributions. Let $q_{j,k} = q_{k,j}$ be the rate at which a particle goes from j to an unoccupied site k. Put

$$\mathcal{H} = \Big\{ \alpha : S \to [0,1], \sum_k q_{j,k} [\alpha(k) - \alpha(j)] = 0 \Big\}.$$

For $\alpha \in \mathcal{H}$, let ν_α be the product measure with

$$\nu_\alpha \{ \eta : \eta(j) = 1 \} = \alpha(j).$$

Then $\mu_\alpha = \lim_{t \to \infty} \nu_\alpha T(t)$ is strong Rayleigh. This is the most general extremal stationary distribution for the process.

Limit Theorems

Proposition. *If $\{\eta(k)\}$ is strong Rayleigh, then there exist independent $\{\zeta(j)\}$ so that $\sum \eta(k)$ and $\sum \zeta(j)$ have the same distribution.*

Proof. Put $N = \sum_k \eta(k)$. Then $f(z, z, ..., z) = E^\mu z^N \neq 0$ if z has positive imaginary part. Therefore,

$$f(z, z, ..., z) = \prod_i [p_i z + (1 - p_i)] = E z^M, \quad M = \sum_j \zeta(j).$$

Theorem. *Suppose the Bernoulli random variables $\{\eta_n(x)\}$ are strong Rayleigh for each n.*

(a) If $\lim_{n\to\infty} \sum_x E\eta_n(x) = \lambda$, $\lim_{n\to\infty} \sum_x [E\eta_n(x)]^2 = 0$, and

$$\lim_{n\to\infty} \sum_{x\neq y} Cov(\eta_n(x), \eta_n(y)) = 0,$$

then

$$\sum_x \eta_n(x) \Rightarrow Poisson(\lambda).$$

(b) If $\lim_{n\to\infty} Var(\sum_x \eta_n(x)) = \infty$, then

$$\frac{\sum_x \eta_n(x) - E\sum_x \eta_n(x)}{\sqrt{Var(\sum_x \eta_n(x))}} \Rightarrow N(0, 1).$$

Applications

1. Take $S = Z^1$, $q_{j,k} = p(j-k)$. Let

$$\eta_0(k) = \begin{cases} 1 & \text{if } k \le 0, \\ 0 & \text{if } k > 0, \end{cases}$$

and $W_t = \sum_{x>0} \eta_t(x)$.

Theorem. *Suppose* $\sigma^2 = \sum_n n^2 p(n) < \infty$. *Then*

$$\frac{W_t - EW_t}{[Var(W_t)]^{1/2}} \Rightarrow N(0,1),$$

$$\lim_{t\to\infty} \frac{EW_t}{\sqrt{t}} = \frac{\sigma}{\sqrt{2\pi}}, \quad \textit{and} \quad \frac{Var(W_t)}{t^{1/2}}$$

is asymptotically between two positive constants.

2. Suppose S is the the binary tree, and $q_{j,k} = \frac{1}{3}$ if $d(j,k) = 1$.

Theorem. *For a natural choice of* α, *with respect to* μ_α,

$$\sum_{k\in L: l(k)=n} \eta(k) \Rightarrow \textit{ Poisson } (1/3), \quad \textit{and}$$

$$\sigma_n^{-1}\left[\sum_{k\in L:l(k)<n} \eta(k) - \frac{n}{3}\right] \Rightarrow N(0,1)$$

with $\frac{23}{189} \le \sigma_n^2/n \le \frac{1}{3}$ *asymptotically.*