The Exclusion Process: Central Limit Theorems and Stationary Distributions

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The exclusion process. S is countable, and $p(x, y)$ are the transition probabilities for an irreducible discrete time Markov chain on S:

$$p(x, y) \geq 0 \text{ and } \sum_y p(x, y) = 1.$$

The exclusion process is a continuous time Markov process η_t on $\{0, 1\}^S$ in which a particle at x waits a unit exponential time, and then tries to move to y with probability $p(x, y)$. If y is vacant, it moves to y, while if y is occupied, it stays at x.

Pemantle's problem (2000). Suppose

$$S = Z^1 \quad \text{and} \quad p(x, x+1) = p(x, x-1) = \frac{1}{2}.$$

At $t = 0$, take

$$\eta = \cdots 1\ 1\ 1\ 0\ 0\ 0\ \cdots,$$

and let

$$N_t = \sum_{x>0} \eta_t(x).$$

Is it true that

$$\frac{N_t - EN_t}{[\text{var}(N_t)]^{1/2}} \Rightarrow N(0,1)?$$

The difficulty: N_t is a sum of Bernoulli random variables, but they are NOT independent. In fact, they are negatively correlated. This leads to a general question: If $p(x,y) = p(y,x)$ and the initial distribution is deterministic (or a product measure), what can be said about the distribution of η_t?

The **generating polynomial** of a probability measure μ on $\{0,1\}^n$ is

$$f(z_1,...,z_n) = E^\mu \prod_{k=1}^n z_k^{\eta(k)}.$$

μ is said to be **stable** if $f \neq 0$ whenever

$$Im(z_k) > 0 \text{ for } 1 \leq k \leq n.$$

Example. If $\mu = \nu_\alpha$ is the product measure with marginals

$$\nu_\alpha\{\eta : \eta(k) = 1\} = \alpha_k,$$

then

$$f(z_1,...,z_n) = \prod_{k=1}^n \Big[\alpha_k z_k + (1-\alpha_k)\Big],$$

so product measures are stable.

Theorem 1 *For a symmetric exclusion process, if the initial distribution is stable, then so is the distribution at later times.*

Theorem 2 *If the distribution of*

$$\{\eta(k), 1 \leq k \leq n\}$$

is stable, then there exist independent Bernoulli random variables

$$\{\zeta(k), 1 \leq k \leq n\}$$

so that

$$\sum_k \eta(k) \text{ and } \sum_k \zeta(k)$$

have the same distribution.

To see this, note that

$$f(z, ..., z) = Ez^{\sum_k \eta(k)} = \sum_{j=0}^{n} P\Big(\sum_k \eta(k) = j\Big) z^j$$

is not zero if $Im(z) > 0$ or if $Im(z) < 0$ or if $z > 0$, so all roots are negative:

$$Ez^{\sum_i \eta(i)} = \prod_{k=1}^{n} \Big[\alpha_k z + (1 - \alpha_k)\Big],$$

where the roots are $-(1 - \alpha_k)/\alpha_k$.

Preservation of stability by symmetric exclusion:

It is enough to check it for exclusion on two sites, i.e., to check that stability is preserved by the transformation

$$\mu \to T\mu = p\mu + (1-p)\mu_{k,l},$$

where $\mu_{k,l}$ is obtained from μ by permuting $\eta(k)$ and $\eta(l)$.

Suppose f is stable. Need to show that

$$Tf(z) \neq 0 \text{ if } Im(z_j) > 0 \text{ for all } j.$$

Fix z_j for $j \neq k, l$. Need to show that T preserves stability of polynomials of the form

$$h(z,w) = a + bz + cw + dzw,$$

where a, b, c, d are **complex**. Such an h is stable iff

$$Re(b\overline{c} - a\overline{d}) \geq |bc - ad|,$$

$$Im(a\overline{b}) \geq 0, Im(a\overline{c}) \geq 0, Im(b\overline{d}) \geq 0, Im(c\overline{d}) \geq 0.$$

Back to **Pemantle's problem**:

By the Lindeberg-Feller Theorem, it is enough to consider second moments. By duality,

$$EN_t = EX_t^+$$

and

$$\sum_{x>0} E\eta_t(x)^2 = E\min(X_t^+, Y_t^+),$$

where X_t and Y_t are independent simple random walks on Z^1 starting at 0. It is harder to estimate the sum of covariances,

$$\sum_{x,y>0,x\neq y} \mathrm{cov}(\eta_t(x), \eta_t(y)).$$

But this can be done, with the result that

$$\lim_{t\to\infty} \frac{EN_t}{\sqrt{t}} = \frac{1}{\sqrt{2\pi}}$$

and

$$0 < c_1 \le \frac{\mathrm{var}(N_t)}{\sqrt{t}} \le c_2 < \infty.$$

It follows that the central limit theorem for N_t holds.

Stationary distributions. From now on, take $S = Z^d$ and $p(x, y) = p(y - x)$. Then the homogeneous product measures

$$\nu_\rho, \quad 0 \leq \rho \leq 1$$

are stationary. Main questions: Are there other (extremal) stationary distributions? If so, what are they?

1. No if $p(-x) = p(x)$, or if

$$d = 1 \text{ and } \sum_x xp(x) = 0,$$

Open problem: How about $d > 1$ and

$$\sum_x xp(x) = 0?$$

2. Suppose $d = 1$, $p(1) = p > p(-1) = q$ and $p(x) = 0$ if $|x| > 1$. Then the inhomogeneous product measure ν_α with

$$\alpha(x) = \frac{p^x}{p^x + q^x}$$

is stationary.

This measure is not extremal. To see this, let

$$C = \bigcup_{n=-\infty}^{\infty} C_n,$$

where

$$C_n = \left\{\eta : \sum_{x<n} \eta(x) = \sum_{x\geq n} [1-\eta(x)] < \infty\right\}.$$

Then η_t is an irreducible Markov chain on each C_n, and $\nu_\alpha(C) = 1$. Therefore, η_t restricted to C_n is positive recurrent with unique stationary distribution $\mu_n(\cdot) = \nu_\alpha(\cdot \mid C_n)$. The extremal stationary distributions in this case are exactly

$$\{\nu_\rho, 0 \leq \rho \leq 1\} \cup \{\mu_n, n \in Z^1\}.$$

Terminology: (i) A measure μ satisfying $\mu(C) = 1$ is said to be blocking. (ii) If it satisfies the weaker conditions

$$\lim_{x\to-\infty} \mu\{\eta : \eta(x) = 1\} = 0,$$
$$\lim_{x\to\infty} \mu\{\eta : \eta(x) = 1\} = 1,$$

it is said to be profile.

3. Suppose $d = 1$ and $\sum_x xp(0,x) > 0$. Then:

(a) The extremal stationary distributions are either (i) $\{\nu_\rho, 0 \le \rho \le 1\}$ or

(ii) $$\{\nu_\rho, 0 \le \rho \le 1\} \cup \{\mu_n, n \in Z^1\},$$

where μ_n are profile measures, and are shifts of each other.

(b) If $p(\cdot)$ has finite range, or satisfies

$$\sum_{x<0} x^2 p(x) < \infty$$

and some reasonable monotonicity conditions, then (ii) holds and μ_n is blocking.

(c) If $\sum_{x<0} x^2 p(x) = \infty$, then there are no stationary blocking measures.

Open problem: In case (c) above, are there stationary profile measures?

4. Suppose $S = Z^2$ and

$$\begin{array}{ccc} & p_2 & \\ & \uparrow & \\ q_1 \leftarrow & x & \rightarrow p_1 \\ & \downarrow & \\ & q_2 & \end{array}$$

with $p_1 > q_1, p_2 > q_2$, and $v \in Z^2$.

(a) There are v-profile stationary **product** measures if and only if v is one of

$$(1,0),\ (0,1),\ \text{or}\ \left(\log \frac{p_1}{q_1}, \log \frac{p_2}{q_2}\right).$$

(b) There is no v-profile stationary measure if $< m, v > \leq 0$, where $m = (p_1 - q_1, p_2 - q_2)$, is the mean vector.

Open problem: What if $< m, v >> 0$?