Two Problems on Stirring Processes –

And their Solutions

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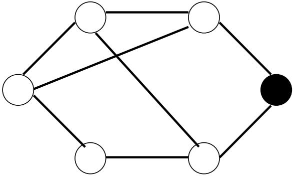
Given a graph $G = (V, E)$, associate with each edge $e \in E$ a Poisson process Π_e with rate $c_e \geq 0$.

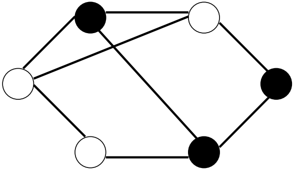
Labels are put on the vertices $v \in V$. At the event times of Π_e, interchange the contents of the two vertices joined by e.

Depending on the nature of the labels, one can define various Markov chains:

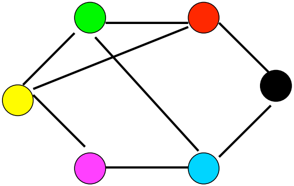
One particle MC



Symmetric exclusion



MC on permutations



For the Markov chain on permutations, think in terms of

Card Shuffling

Vertices = positions in a deck	**labels = cards**
1	Three of Clubs
2	Two of Diamonds
$\vdots$	$\vdots$
i	Five of Diamonds
$\vdots$	$\vdots$
j	Ten of Spades
$\vdots$	$\vdots$
52	Jack of Diamonds

At rate $c_{i,j}$, interchange **Five of Diamonds** and **Ten of Spades**.

Pemantle's problem (2000). Suppose

$$G = Z^1 \quad \text{and} \quad c_e = \frac{1}{2} \text{ for each } e.$$

Symmetric exclusion: At $t = 0$, take

$$\eta = \cdots 1\ 1\ 1\ 0\ 0\ 0\ \cdots,$$

and let

$$N_t = \sum_{x>0} \eta_t(x) = \#\text{ particles to the right of the origin at time } t.$$

Question: Does N_t satisfy the Central Limit Theorem?

The difficulty: N_t is a sum of Bernoulli random variables, but they are NOT independent. In fact, they are negatively correlated. This leads to a question for general G:

If the initial distribution is deterministic (or a product measure), what can be said about the distribution at time t?

The **generating polynomial** of $\{\eta(1), ..., \eta(n)\}$ is

$$f(z_1, ..., z_n) = E^\mu \prod_{k=1}^{n} z_k^{\eta(k)}.$$

It is said to be **stable** if $f \neq 0$ whenever

$$Im(z_k) > 0 \text{ for } 1 \leq k \leq n.$$

Example. If $\eta(k)$ are independent with

$$P(\eta(k) = 1) = p_k,$$

then

$$f(z_1, ..., z_n) = \prod_{k=1}^{n} \left[p_k z_k + (1 - p_k)\right],$$

so independent Bernoullis are stable.

Connection with negative correlations:

Theorem

If the distribution of

$$\eta = \{\eta(k), 1 \leq k \leq n\}$$

is stable, then the random variables are negatively associated, in the sense that

$$Ef(\eta)g(\eta) \leq Ef(\eta)Eg(\eta)$$

for all $f, g \uparrow$ *depending on disjoint sets of variables.*

Theorem

For a symmetric exclusion process, if the initial distribution is stable, then so is the distribution at later times.

(Based on work with J. Borcea and P. Branden)

Theorem

If the distribution of

$$\{\eta(k), 1 \le k \le n\}$$

is stable, then there exist independent Bernoulli random variables

$$\{\zeta(k), 1 \le k \le n\}$$

so that

$$\sum_k \eta(k) \quad \text{and} \quad \sum_k \zeta(k)$$

have the same distribution.

For the second result, let $N = \sum_k \eta(k)$, and note that

$$f(z, ..., z) = Ez^N = \sum_{j=0}^{n} P\big(N = j\big) z^j$$

is not zero if $Im(z) > 0$ or if $Im(z) < 0$ or if $z > 0$, so all roots are negative:

$$Ez^N = \prod_{k=1}^{n} \big[\alpha_k z + (1 - \alpha_k)\big],$$

where the roots are $-(1 - \alpha_k)/\alpha_k$.

Preservation of stability by symmetric exclusion:
It is enough to check it for exclusion on two sites, i.e., to check that stability is preserved by the transformation

$$\mu \to T\mu = p\mu + (1-p)\mu_{k,l},$$

where $\mu_{k,l}$ is obtained from μ by permuting $\eta(k)$ and $\eta(l)$. Suppose f is stable. Need to show that

$$Tf(z) \neq 0 \text{ if } Im(z_j) > 0 \text{ for all } j.$$

Fix z_j for $j \neq k, l$. Need to show that T preserves stability of polynomials of the form

$$h(z,w) = a + bz + cw + dzw,$$

where a, b, c, d are **complex**. Such an h is stable iff

$$Re(b\bar{c} - a\bar{d}) \geq |bc - ad|,$$

$$Im(a\bar{b}) \geq 0, Im(a\bar{c}) \geq 0, Im(b\bar{d}) \geq 0, Im(c\bar{d}) \geq 0.$$

Back to **Pemantle's problem:**

By the Lindeberg-Feller Theorem, it is enough to consider the first two moments. By duality,

$$EN_t = \sum_{x>0} E\eta_t(x) = EX_t^+$$

and

$$\sum_{x>0}[E\eta_t(x)]^2 = E\min(X_t^+, Y_t^+),$$

where X_t and Y_t are independent simple random walks on Z^1 starting at 0. It is harder to estimate the sum of covariances,

$$\sum_{x,y>0,x\neq y} \mathrm{cov}(\eta_t(x), \eta_t(y)).$$

But this can be done, with the result that

$$\lim_{t \to \infty} \frac{EN_t}{\sqrt{t}} = \frac{1}{\sqrt{2\pi}}$$

and

$$0 < c_1 \leq \frac{\mathrm{var}(N_t)}{\sqrt{t}} \leq c_2 < \infty.$$

Theorem

$$\frac{N_t - EN_t}{[var(N_t)]^{1/2}} \Rightarrow N(0, 1).$$

Aldous' conjecture (1992). Let Q be the rate matrix for a symmetric, irreducible n-state Markov chain, i.e., $q_{i,j}$ is the exponential rate at which the chain goes from state i to state j. Then $-Q$ has eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}.$$

The smallest positive eigenvalue λ_1 determines the rate of convergence to equilibrium:

$$p_t(i,j) = \frac{1}{n} + a_{i,j} e^{-\lambda_1 t} + o(e^{-\lambda_1 t}).$$

Consider the stirring process on the complete graph G with n vertices. The one particle Markov chain has $q_{i,j} = c_{i,j}$. The Markov chain on permutations has $q_{\pi,\pi_{i,j}} = c_{i,j}$, where $\pi_{i,j}$ is the permutation obtained from π by applying the transposition interchanging i and j.

Let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}.$$

be the eigenvalues for the one particle Markov chain, and

$$0 = \lambda_0^* < \lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_{n!-1}^*.$$

be the eigenvalues for the Markov chain on permutations of the vertices.

Each $\lambda_i =$ some λ_j^*, so $\lambda_1^* \leq \lambda_1$.

In fact, any sum of the form

$$\lambda_{i_1} + \cdots + \lambda_{i_k}$$

is an eigenvalue of the permutation chain. There are $\sim 4^{n-1}/\sqrt{\pi n}$ eigenvalues of this type, and all are $\geq \lambda_1$. How about the others?

Aldous' Conjecture (1992): $\lambda_1^* = \lambda_1$.

Why guess this?

1. True for $c_e \equiv 1$ on complete graph – Diaconis and Shashahani (1981).
2. True for $c_e \equiv 1$ on star graphs – Flatto, Odlyzko and Wales (1985).
3. True for general c_e on trees – Handjani and Jungreis (1996).
4. True for $c_e \equiv 1$ on complete multipartite graphs – Cesi (2009).
5. Other related results by Koma and Nachtergele (1997), Morris (2008), Starr and Conomos (2008), and Dieker (2009).

Why should you care?

1. It is MUCH easier to compute eigenvalues for an $n \times n$ matrix than for an $n! \times n!$ matrix.
2. "Intermediate" chains, such as symmetric exclusion have the same smallest eigenvalue.

Theorem

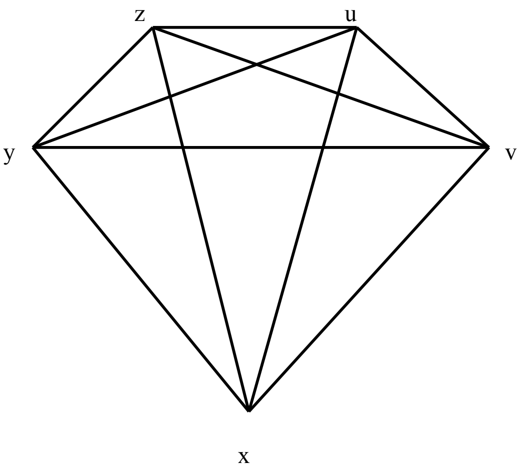
For arbitrary rates, $\lambda_1^* = \lambda_1$.

(Joint with P. Caputo and T. Richthammer)

The proof is inductive. Remove vertex x and edges leading to it. The rates for the remaining edges are increased:

$$\text{New } c_{\{y,z\}} = c_{\{y,z\}} + \frac{c_{\{x,y\}} c_{\{x,z\}}}{c_x}; \quad c_x = \sum_{y \neq x} c_{\{x,y\}}.$$

For the inductive step to work, need to check that a certain $n! \times n!$ matrix C is positive semi-definite.


z
u
y
v
x

If $n = 3$, for example,

$$C = \begin{pmatrix} c & 0 & 0 & -c_1 d & -c_2 d & c_1 c_2 \\ 0 & c & 0 & -c_2 d & c_1 c_2 & -c_1 d \\ 0 & 0 & c & c_1 c_2 & -c_1 d & -c_2 d \\ -c_1 d & -c_2 d & c_1 c_2 & c & 0 & 0 \\ -c_2 d & c_1 c_2 & -c_1 d & 0 & c & 0 \\ c_1 c_2 & -c_1 d & -c_2 d & 0 & 0 & c \end{pmatrix},$$

where $c = c_1^2 + c_1 c_2 + c_2^2$ and $d = c_1 + c_2$. The eigenvalues of C in this case are 0 and $2c$, each with multiplicity 3.

Idea: Try to write C as the covariance matrix of $Z = (X, Y)$, where X and Y are $n!/2$ random vectors. Choose X to have iid components with variance c, and then $Y = AX$, where A is chosen so that $cov(X, Y)$ is right. Then hope that the components of Y are uncorrelated and have variance c. This works for $n = 3$, but fails for larger n.

However: It turns out that $cov(Y) \leq cI$, which is all that is needed.

To check this, write $cI - cov(Y)$ as a linear combination of matrices A_i; the coefficients are products of rates, but A_i does not depend on the rates.

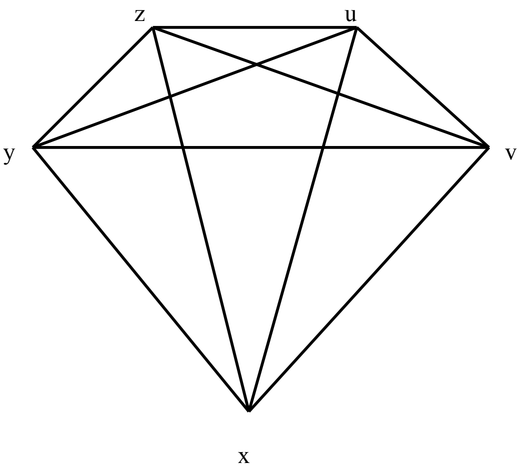
Need to know that certain sums and differences of the A_i's are positive semi-definite.

For example for $n = 5$, there is a certain 60×60 matrix B with small integer entries that must be considered. It turns out that $B^2 = 24B$, so its only eigenvalues are 0 and 24. In fact, the multiplicities are 45 and 15 respectively.

But what about larger n? It turns out that the corresponding matrix has a block form:

$$\begin{matrix} B & 0 & 0 & \cdots \\ 0 & B & 0 & \cdots \\ 0 & 0 & B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Why is $n = 5$ the main case? Each transposition affects two vertices, and we are looking at the square of a matrix, so entries in the square involve at most four vertices. But then there is the special vertex x that was removed in the induction argument, for a total of 5.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Back to stability

For the random cluster model:

$q \geq 1$: Associated by FKG Theorem.

$q < 1$: Stable iff the graph is a tree. Is it negatively associated in general? Who knows?