Approximating Multiples of Strong Rayleigh Random Variables

Thomas M. Liggett, UCLA

Consider a polynomial with positive coefficients

$$f(u) = \sum_{k=0}^{n} c_k u^k, \quad c_k > 0.$$

It is said to be Strong Rayleigh (SR) if all of its roots are real (and hence negative). A random variable X taking values $0, 1, \ldots, n$ is SR if its probability generating polynomial (pgf)

$$f(u) = Eu^X = \sum_{k=0}^{n} P(X = k) u^k$$

is SR.

In this case,

$$f(u) = \prod_{k=0}^{n} [p_k u + (1 - p_k)]$$

Therefore

$$X =_d \eta_1 + \cdots + \eta_n,$$

where η_i are independent Bernoulli random variables with parameters p_k. If X_n is a sequence of SR random variables, this gives a triangular array

$$\begin{aligned} X_1 =& \eta_{1,1} \\ X_2 =& \eta_{2,1} + \eta_{2,2} \\ X_3 =& \eta_{3,1} + \eta_{3,2} + \eta_{3,3}, \ldots \end{aligned}$$

of Bernoulli random variables with independence in each row. It follows from the Lindeberg-Feller Theorem that if $var(X_n) \to \infty$, X_n satisfies the CLT.

Definition A random vector $\mathbf{X}$ is said to be SR if its pgf $f(\mathbf{u}) \neq 0$ whenever $Im(u_i) > 0$ for all i.

Many natural distributions satisfy SR. But even if one does not know the distribution of $\mathbf{X}$ explicitly, sometimes SR can be verified indirectly.

For example, consider the exclusion process, which is a Markov process on the state space $\{0, 1\}^S$, where S is a countable set. Let $p(x, y)$ be the transition probabilities for a Markov chain on S. Each particle has a rate 1 exponential clock. When the clock at x rings, if the there is a particle at x, it tries to move to y with probability $p(x, y)$. If y is occupied, it stays at x; otherwise it moves to y.

The process is said to be symmetric if $p(x, y) = p(y, x)$ for all x, y.

One of many questions about it is the following:

Defiinition A probability measure on $\{0,1\}^S$ is said to be negatively associated if $f(\eta)$ and $g(\eta)$ are negatively correlated for all increasing functions f, g that depend on disjoint sets of coordinates.

Problem Is It the case that η_t is negatively associated whenever η_0 is?

Answer No, even in the symmetric case.

However, in

J. Borcea, P. Brändén and T. Liggett. Negative dependence and the geometry of polynomials. *JAMS* **22** (2009) 521–567, we proved that the symmetric exclusion interacting particle system $\eta_t \in \{0,1\}^S$ satisfies the following property:

$$\eta_0 \quad SR \quad \Rightarrow \quad \eta_t \quad SR.$$

Moreover SR implies negative association.

Using this,

T. Liggett. Distributional limits for the symmetric exclusion process. Stoch. Proc. Appl. **119** (2009) 1–15

proved CLT's for the symmetric exclusion process.

In

S. Ghosh, T. Liggett and R. Pemantle. Multivariate CLT follows from strong Rayleigh property. ANALCO17 (2017) 139–147,

we raised the question of the extent to which SR implies a multivariate CLT. This is quite different from the univariate case, since the pgf no longer factors, and there is no reason to think that $\mathbf{X}$ can be written as a sum of independent random vectors.

Using a result of Lebowitz, Pittel, Ruelle and Speer, we did prove such a result, but with the assumption $var(\mathbf{X_n}) >> n^{\frac{1}{3}}$.

Why do we need a stronger assumption in the multivariate case?

Deducing multivariate CLT's from univariate CLT's via the Cramér-Wold device:

$$\mathbf{X}_n \to_d \mathbf{X} \quad \text{iff} \quad \mathbf{b} \cdot \mathbf{X}_n \to_d \mathbf{b} \cdot \mathbf{X}$$

for every $\mathbf{b}$.

This is a simple consequence of the fact that distributional convergence is equivalent to convergence of the characteristic functions (=Fourier transforms).

Problem: If X is SR, bX is not even integer valued, much less SR. Can bX be well approximated by a SR random variable?

Ghosh, Liggett and Pemantle (2017) proved that

$$\text{if } X \text{ is } SR, \text{ then } \left\lfloor \frac{1}{k} X \right\rfloor \text{ is } SR.$$

However, if X is $B(3n, \frac{1}{2})$, then the roots z_i of the pgf of $\lfloor \frac{2}{3} X \rfloor$ satisfy

$$2 \max_i [Im(z_i)]^2 \geq 9n^2 - 9n - 1.$$

Maybe $\lfloor \frac{j}{k} X \rfloor$, should be written as a sum of independent random variables with more than 2 values....

Theorem. If X is SR, the pgf of $\lfloor \frac{2}{k} X \rfloor$ can be factored into quadratic polynomials with positive coefficients, so $\lfloor \frac{2}{k} X \rfloor$ has the same distribution as the sum of independent random variables taking the values 0,1,2.

Definition f has property P_j if it can be factored into polynomials of degree at most j with positive coefficients.

$$P_1 \iff SR \iff \text{ all roots real.}$$

$$P_2 \iff \text{Hurwitz} \iff \text{ all roots have negative real part.}$$

P_3 is not a statement about each root. If f is P_3 but not P_2, each root z with positive real part must be paired with a negative root w so that

$$2Re(z) < -w < |z|^2/2Re(z).$$

Theorem (Hermite-Bieler) Write

$$f(u) = \sum_{m=0}^{1} u^m h_m(u^2) = h_0(u^2) + uh_1(u^2).$$

Then f is P_2 iff the roots of h_0, h_1 are negative and simple and interlace, with the largest being a root of h_0.

Definition f has property Q_j if writing

$$f(u) = \sum_{m=0}^{j-1} u^m h_m(u^j),$$

the roots of $h_0, h_1, \ldots, h_{j-1}$ are negative and simple and interlace, with the largest being a root of h_0.

Note that $Q_1 = P_1, Q_2 = P_2$. However, neither implication between Q_3 and P_3 is true.

Location of roots
If f is P_3, it has no roots in the sector

$$\{z : Re(z) > 0, (Im(z))^2 \leq 3(Re(z))^2\}.$$

If f is Q_3, it has no roots on

$$\{z : Re(z) > 0, (Im(z))^2 = 3(Re(z))^2\}.$$

Theorem If X is SR, then $\lfloor \frac{j}{k} X \rfloor$ is Q_j.

Corollary If X is SR, then $\lfloor \frac{2}{k} X \rfloor$ is P_2.

Conjecture If X is SR, then $\lfloor \frac{3}{4} X \rfloor$ is P_3.

This is true if $X \leq 6$. If X is $B(40, p)$ with $p = \frac{1}{8}, \frac{1}{4}$ or $\frac{1}{2}$, the pgf of $\lfloor \frac{3}{4} X \rfloor$ has 10 real roots

$$w_{10} < w_9 < \cdots < w_1 < 0$$

and 10 conjugate pairs of roots

$$z_1, \bar{z}_1, \ldots, z_{10}, \bar{z}_{10}$$

with $0 < Re(z_1) < \cdots < Re(z_{10})$. With this ordering, $(u - w_i)(u - z_i)(u - \bar{z}_i)$ has positive coefficients for each $1 \leq i \leq 10$.

If X is $B(21, \frac{1}{2})$, $\lfloor \frac{3}{5} X \rfloor$ is not P_3. However, it is almost P_3 in the sense that its pgf is the product 4 cubics, only one of which has a negative coefficient:

$$22(.00031 + .021u - .0058u^2 + u^3)(.12 + .43u + .14u^2 + u^3)$$

$$\cdot(8.96 + 5.88u + .92u^2 + u^3)(2993 + 317u + 8.49u^2 + u^3).$$

The same pattern occurs if X is $B(n, \frac{1}{2})$ for $n = 25, 35, 50$.

Proposition If U, V are nonnegative integer valued random variables whose pgf's satisfy

$$Eu^U = Eu^V(au^3 + bu^2 + cu + d)$$

with $d \geq 0, c + d \geq 0, b + c + d \geq 0$, then there is a coupling so that $U \leq V + 3$ and $E(V + 3 - U) = b + 2c + 3d$.

The underlying fact that our results depend on involves polynomials with interlacing roots:

Theorem (Ghosh, Liggett, Pemantle). Let f be the pgf of a SR X taking values $0, 1, \ldots, n$, which is a polynomial of degree n with all negative roots. Write

$$f(x) = \sum_{i=0}^{k-1} x^i g_i(x^k),$$

where g_i is a polynomial of degree $\lfloor \frac{n-i}{k} \rfloor$. Then $g_0, g_i, \ldots, g_{k-1}$ have interlacing, negative simple roots, with the largest being a root of g_0.

The proof is by induction on the degree of f.

For the induction argument, write $F(x) = (x + r)f(x)$ with $r > 0$, where f has degree n and F has degree $n + 1$. Consider the corresponding decomposition for F:

$$F(x) = \sum_{i=0}^{k-1} x^i G_i(x^k).$$

Then

$$G_i(y) = rg_i(y) + \begin{cases} yg_{k-1}(y) & \text{if } i = 0; \\ g_{i-1}(y) & \text{if } i \geq 1. \end{cases}$$

Let the roots of the g_i's be $\cdots < s_4 < s_3 < s_2 < s_1 < s_0 < 0$.

Then for $k = 3$, for example, the following explains the proof.

$$\begin{pmatrix} & \cdots & s_6 & s_5 & s_4 & s_3 & s_2 & s_1 & s_0 & 0 \\ \\ g_0 & \cdots & 0 & + & + & 0 & - & - & 0 & + \\ g_1 & \cdots & + & + & 0 & - & - & 0 & + & + \\ g_2 & \cdots & + & 0 & - & - & 0 & + & + & + \\ \\ G_0 & \cdots & - & + & + & + & - & - & - & + \\ G_1 & \cdots & + & + & + & - & - & - & + & + \\ G_2 & \cdots & + & + & - & - & - & + & + & + \end{pmatrix}.$$

So, G_0 has a root in $\ldots, (s_3, s_2), (s_0, 0)$, G_1 has a root in $\ldots, (s_4, s_3), (s_1, s_0)$, and G_2 has a root in $\ldots, (s_5, s_4), (s_2, s_1)$.

The proof that $\lfloor \frac{j}{k} X \rfloor$ is Q_j if X is SR is similar. The h_i's in the definition of property Q_j are

$$h_i(u) = \sum_{ik \le mj < (i+1)k} g_m(u).$$

For $j = 4, k = 7$

$$h_0 = g_0 + g_1, \quad h_1 = g_2 + g_3, \quad h_2 = g_4 + g_5, \quad h_3 = g_6.$$

The proof is described in the following form:

$$
\begin{pmatrix}
 & \cdots & s_7 & s_6 & s_5 & s_4 & s_3 & s_2 & s_1 & s_0 \\
 \\
g_0 & \cdots & 0 & - & - & - & - & - & - & 0 \\
g_1 & \cdots & - & - & - & - & - & - & 0 & + \\
g_2 & \cdots & - & - & - & - & - & 0 & + & + \\
g_3 & \cdots & - & - & - & - & 0 & + & + & + \\
g_4 & \cdots & - & - & - & 0 & + & + & + & + \\
g_5 & \cdots & - & - & 0 & + & + & + & + & + \\
g_6 & \cdots & - & 0 & + & + & + & + & + & + \\
 \\
h_0 & \cdots & - & - & - & - & - & - & - & + \\
h_1 & \cdots & - & - & - & - & - & + & + & + \\
h_2 & \cdots & - & - & - & + & + & + & + & + \\
h_3 & \cdots & - & 0 & + & + & + & + & + & +
\end{pmatrix}
$$