A Proof of Aldous' Spectral Gap Conjecture

Thomas M. Liggett, UCLA

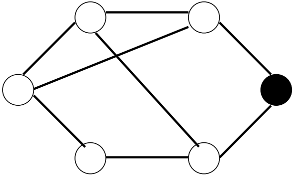
Stirring Processes

Given a graph $G = (V, E)$, associate with each edge $e \in E$ a Poisson process Π_e with rate $c_e \geq 0$.

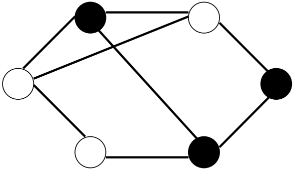
Labels are put on the vertices $v \in V$. At the event times of Π_e, interchange the contents of the two vertices joined by e.

Depending on the nature of the labels, one can define various continuous time Markov chains:

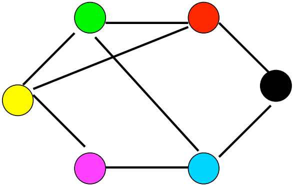
One particle MC



Symmetric exclusion



MC on permutations



Let Q be the rate matrix for a symmetric, irreducible n-state Markov chain, i.e., $q(x,y) = q(y,x)$ is the exponential rate at which the chain goes from state x to state y:

$$p_t(x,y) = q(x,y)t + o(t), \quad t \downarrow 0 \text{ for } x \neq y.$$

Then $-Q$ has eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}.$$

The smallest positive eigenvalue λ_1 determines the rate of convergence to equilibrium:

$$p_t(x,y) = \frac{1}{n} + a(x,y)e^{-\lambda_1 t} + o(e^{-\lambda_1 t}), \quad t \uparrow \infty.$$

It is the largest value of λ for which

$$\frac{1}{2}\sum_{x,y} q(x,y)[g(y) - g(x)]^2 \geq \lambda \sum_x g(x)^2, \quad \sum_x g(x) = 0.$$

Consider the stirring process on the complete graph G with n vertices. The one particle Markov chain has $q(x, y) = c_{xy}$. The Markov chain on permutations has $q(\pi, \pi_{xy}) = c_{xy}$, where π_{xy} is the permutation obtained from π by applying the transposition that interchanges x and y.

Let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}.$$

be the eigenvalues for the one particle Markov chain, and

$$0 = \lambda_0^* < \lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_{n!-1}^*.$$

be the eigenvalues for the Markov chain on permutations.

Note: Each $\lambda_i =$ some λ_j^*, so $\lambda_1^* \leq \lambda_1$.

In fact, any sum of the form

$$\lambda_{i_1} + \cdots + \lambda_{i_k}$$

is an eigenvalue of the permutation chain. There are $\sim 4^{n-1}/\sqrt{\pi n}$ eigenvalues of this type, and all are $\geq \lambda_1$. How about the others?

Aldous' Conjecture (1992): $\lambda_1^* = \lambda_1$.

Why guess this?

1. True for $c_e \equiv 1$ on complete graph – Diaconis and Shahshahani (1981).
2. True for $c_e \equiv 1$ on star graphs – Flatto, Odlyzko and Wales (1985).
3. True for general c_e on trees – Handjani and Jungreis (1996).
4. True for $c_e \equiv 1$ on complete multipartite graphs – Cesi (2009).
5. Other related results by Koma and Nachtergele (1997), Morris (2008), Starr and Conomos (2008), and Dieker (2009).

Theorem.

On a general finite graph with arbitrary rates,

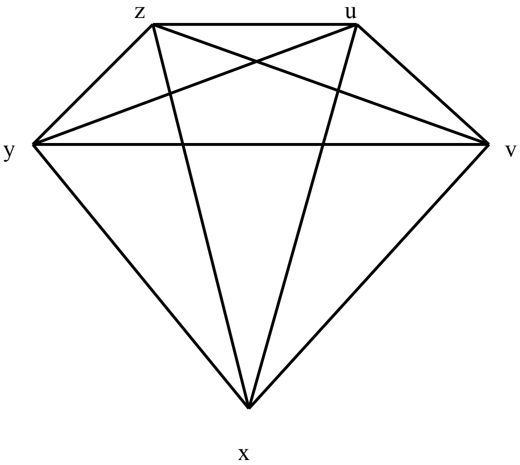
$$\lambda_1^* = \lambda_1.$$

(Joint with P. Caputo (Rome) and T. Richthammer (UCLA).)

Why should you care?

1. It is MUCH easier to compute eigenvalues for an $n \times n$ matrix than for an $n! \times n!$ matrix.

2. "Intermediate" chains, such as symmetric exclusion have the same smallest positive eigenvalue.


z
u
y
v
x

The proof is by induction on the number of vertices:

Construct G_x by removing vertex x and edges leading to it. The rates for the remaining edges are increased:

$$\text{New } c_{\{y,z\}} = c_{\{y,z\}} + \frac{c_{\{x,y\}} c_{\{x,z\}}}{c_x}; \quad c_x = \sum_{y \neq x} c_{\{x,y\}}.$$

Note:

(a) If x is connected to only two vertices y, z, this corresponds to an electrical network series reduction from $y \leftrightarrow x \leftrightarrow z$ to $y \leftrightarrow z$.

(b) If x is connected to three vertices y, z, w, it corresponds to an electrical network star-triangle reduction from the star with center x to the triangle y, z, w.

(c) The addition at the end corresponds to an electrical network parallel reduction.

Basic steps in the induction argument:

1. $\lambda_1(G_x) \geq \lambda_1(G)$. This is a consequence of the variational characterization: Given a function on G_x, extend it to G by making it harmonic at x.

2. Let $\mathcal{H} = \{f : E[f \mid \text{ position of } i\text{th particle}] = 0 \text{ for each } i\}$, and μ_1^* be the analogue of λ_1^* for functions in $\mathcal{H}$. Then

$$\lambda_1^* = \min\{\lambda_1, \mu_1^*\}.$$

Idea: eigenfunctions $\notin \mathcal{H}$ generate eigenfunctions of the one particle chain.

3. If the octopus inequality holds, then

$$\mu_1^*(G) \geq \max_x \lambda_1^*(G_x).$$

This again uses the variational characterization.

4. $\mu_1^*(G) \geq \max_x \lambda_1^*(G_x) = \max_x \lambda_1(G_x) \geq \lambda_1(G)$. Now use #2.

The **octopus inequality:** For fixed x,

$$\sum_{y\neq x} c_{xy} \sum_{\pi}[f(\pi_{xy})-f(\pi)]^2 \geq \sum_{y,z\neq x} \frac{c_{\{x,y\}}c_{\{x,z\}}}{c_x} \sum_{\pi}[f(\pi_{yz})-f(\pi)]^2.$$

This is equivalent to the positive semi-definiteness of a certain matrix C. If $n=3$, for example,

$$C = \begin{pmatrix} c & 0 & 0 & -c_1 d & -c_2 d & c_1 c_2 \\ 0 & c & 0 & -c_2 d & c_1 c_2 & -c_1 d \\ 0 & 0 & c & c_1 c_2 & -c_1 d & -c_2 d \\ -c_1 d & -c_2 d & c_1 c_2 & c & 0 & 0 \\ -c_2 d & c_1 c_2 & -c_1 d & 0 & c & 0 \\ c_1 c_2 & -c_1 d & -c_2 d & 0 & 0 & c \end{pmatrix},$$

where $c = c_1^2 + c_1 c_2 + c_2^2$ and $d = c_1 + c_2$. The eigenvalues of C in this case are 0 and $2c$, each with multiplicity 3.

Ideas:

1. Try to write C as the covariance matrix of $Z = (X, Y)$, where X and Y are $n!/2$ random vectors. Choose X to have iid components with variance c. How about Y?

2. Try $Y = DX$, where D is chosen so that $cov(X, Y)$ is right. D is unique. Hope that the components of Y are uncorrelated and have variance c. This works for $n = 3$, but fails for $n = 4$.

However: If $n = 4$, it turns out that $cov(Y) \leq cI$, which is all that is needed. Could this be true in general?

To check this, write $cI - cov(Y)$ as a linear combination of matrices A^J; the coefficients (both positive and negative) involve the rates, but the A^J do not. Here $J \subset V$ with $|J| = 4$.

The π, π' entry of A^J is

$$\begin{cases} 2 & \text{if } \pi = \pi' \text{ or } \pi^{-1}\pi' = \text{ a product of 2 disjoint 2-cycles from } J; \\ -1 & \text{if } \pi^{-1}\pi \text{ is a 3-cycle from } J; \\ 0 & \text{otherwise.} \end{cases}$$

Need to know that the A^J's and certain linear combinations B^K of the A^J's are positive semi-definite: For $|K| = 5$ and $x \in K$,

$$B^K = \sum_{J : x \in J \subset K} A^J - A^{K \setminus \{x\}}.$$

Example: For $n = 4$,

$$A = 3\begin{pmatrix} E_4 & 0 & 0 \\ 0 & E_4 & 0 \\ 0 & 0 & E_4 \end{pmatrix} - E_{12},$$

where E_k is the $k \times k$ matrix with all entries $=$1. This A has eigenvalues 0 and 12 with multiplicities 10 and 2 respectively.

Example: For $n = 5$, B is a 60×60 matrix with small integer entries.

It turns out that $B^2 = 24B$, so its only eigenvalues are 0 and 24. In fact, the multiplicities are 45 and 15 respectively.

But what about larger n? It turns out that the corresponding matrices have a block form:

$$\begin{pmatrix} A & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{pmatrix} B & 0 & 0 & \cdots \\ 0 & B & 0 & \cdots \\ 0 & 0 & B & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Here A and B are the matrices from the $n = 4$ and $n = 5$ cases.

The blocks correspond to:

the $n!/4!$ left cosets of the even permutations on 4 sites in the even permutations on n sites for A

and

the $n!/5!$ left cosets of the even permutations on 5 sites in the even permutations on n sites for B.

Why is $n = 5$ the main case?

Each transposition affects two vertices, and we are looking at a matrix of the form $D^t D$, so entries in the product involve at most four vertices. But then there is the special vertex x that was removed in the induction argument, for a total of 5.

Not all Markov chains based on stirring have the same smallest positive eigenvalue.

Example: Perfect matchings – see e.g., Diaconis and Holmes (2002). Take $n = 2k$. At rate c_{xy}:

(a) If x, y are matched, nothing happens.

(b) If not, then x, u and y, v are matched. After the transition, x, v and y, u are matched.

If γ_1 is the smallest positive eigenvalue for this process, then

$$\gamma_1 \geq \lambda_1^* = \lambda_1,$$

but strict inequality can occur. If $k = 2, n = 4$ and $c_e \equiv 1$, $\lambda_1 = 4, \gamma_1 = 6$.