# Stochastic Models for Large Interacting Systems in the Sciences

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*Interacting Particle Systems*, Springer, 1985

*Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Springer, 1999

## (Biased) Voter Models

M. Kimura, *"Stepping stone" model of population*, Ann. Rep. Nat. Inst. Genetics Japan **3** (1953).

T. Williams and R. Bjerknes, *Stochastic model for abnormal clone spread through epithelial basal layer*, Nature **236** (1972).

## Stochastic Ising Models

R. J. Glauber, *Time-dependent statistics of the Ising model*, J. Mathematical Physics **4** (1963).

J. M. Keith, *Segmenting eukaryotic genomes with the generalized Gibbs sampler*, J. Computational Biology **13** (2006).

Kimura won the 1992 Darwin Medal of the Royal Society;
Glauber won the 2005 Nobel Prize in Physics.

## Contact Processes

P. Grassberger and A. de la Torre, *Reggeon field Theory (Schlögl's first model) on a lattice: Monte Carlo calculations of critical behaviour*, Annals of Physics **122** (1979).

R. Schinazi, *On an interacting particle system modeling an epidemic*, J. Mathematical Biology **34** (1996).

## Exclusion Processes

C. T. MacDonald, J. H. Gibbs and A. C. Pipkin, *Kinetics of biopolymerization on nucleic acid templates*, Biopolymers **6** (1968).

H-W. Lee, V. Popkov and D. Kim, *Two-way traffic flow: Exactly solvable model of traffic jam*, J. Physics A **30** (1997).

# The Voter Model

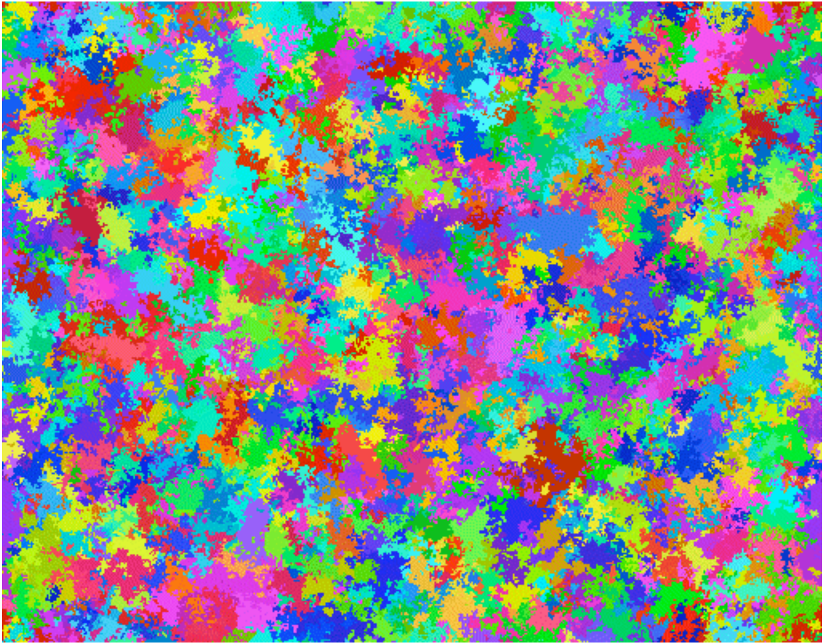
(Holley and Liggett, 1975)

State of the system at any given time: An infinite array of individuals that change opinions at random times:

$$\begin{array}{cccccccc} R & D & R & R & R & D & D & D \\ R & D & D & R & D & R & D & R \\ R & D & R & D & R & D & D & D \\ R & R & D & R & R & R & D & R \\ D & D & R & R & R & D & D & R \\ R & R & R & D & R & D & D & R \\ R & D & D & D & R & D & D & R \\ D & D & R & R & R & D & R & D \end{array}$$

At exponential times, individuals choose a neighbor at random and adopt his/her opinion.

A simulation with many opinions:



Back to two opinions $R$ and $D$. Each individual flips a fair coin to determine his/her initial opinion.

Questions:

(a) What is $P(\text{individual } x \text{ has opinion } R \text{ at time } t)$?

(b) What can be said about

$$P(\text{individuals } x \text{ and } y \text{ have the same opinion at time } t)$$

for large times $t$?

Answers:

(a) $P(\text{individual } x \text{ has opinion } R \text{ at time } t) = \frac{1}{2}$ for all $x, t$.

For question (b), consider the model on a $d$-dimensional array.

If $d = 1$ or $d = 2$,

$$\lim_{t \to \infty} P(\text{individuals } x \text{ and } y \text{ have the same opinion at time } t) = 1.$$

However, if $d \geq 3$, this limit, call it $g(x, y)$, satisfies

$$\frac{1}{2} < g(x, y) < 1 \text{ for } x \neq y$$

and

$$\lim_{y - x \to \infty} g(x, y) = \frac{1}{2}.$$

In fact,

$$g(x, y) = \frac{1}{2} + \frac{1}{2} P(\text{a random walk starting at } y - x \text{ ever hits } 0).$$

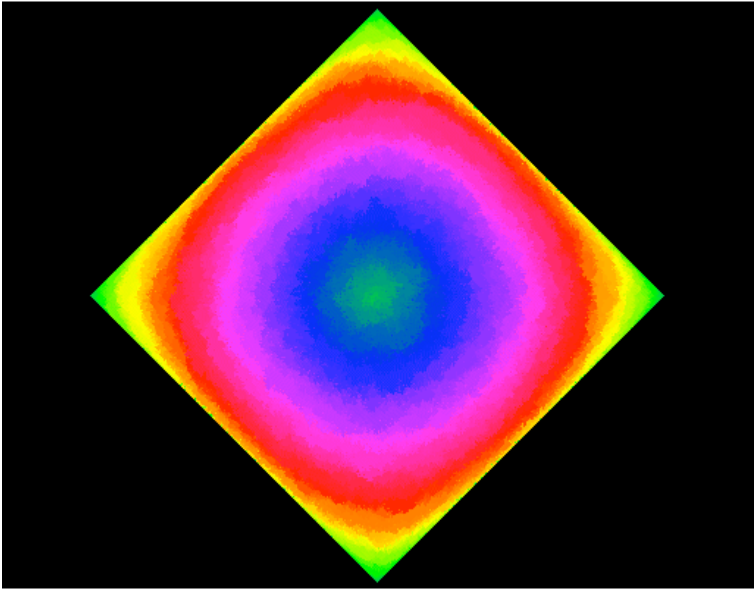
## The Biased Voter Model, or The Williams-Bjerknes Tumor Growth Model (1972)

Differences:
(a) It is more likely that health cells become cancerous than the other way around.
(b) Start with a single cancerous cell. Given that the tumor does not disappear, it has an asymptotic shape.

Here is a simpler discrete time model:

## Richardson's Model (1973)

Cancerous cells remain cancerous. Healthy cells become cancerous at time $n+1$ with probability $p$ if at least one neighbor was cancerous at time $n$.



Question: What is the asymptotic shape?

One answer: It has a straight edge if $p > p_c$, where $p_c$ is the critical value of a discrete time contact process. (Durrett and Liggett, 1981)

Nonrigorously, $p_c \sim .64$. Rigrously, $p_c \leq \frac{2}{3}$.

Question: What is the limiting asymptotic shape as $p \downarrow 0$?

Richardson speculated, based on simulations, that the limiting shape is a circle. Kesten proved that in high dimensions, it is not a sphere.

**Question:** Why consider infinite systems, when real systems are finite?

**Answer:** Infinite models at infinite times are better models for large finite systems at large finite times than are finite models at infinite times.

## The Contact Process

(Harris, 1974)

(a) Infected individuals become healthy at rate 1.
(b) Healthy individuals become infected at rate

$$\lambda \times \#(\text{infected neighbors}).$$

On a finite set, the infection dies out eventually for any $\lambda$.

On a $d$-dimensional grid, there is a critical value $\lambda_d$,

$$\frac{1}{2d} \leq \lambda_d \leq \frac{2}{d},$$

so that

(a) the infection dies out eventually if $\lambda \leq \lambda_d$,
and
(b) survives with positive probability if $\lambda > \lambda_d$.

On an $N \times N \times \cdots \times N$ grid, the extinction time $\tau_N$ satisfies:

$$\tau_N \sim \log N \quad \text{if} \quad \lambda < \lambda_d$$

and

$$\log \tau_N \sim N^d \quad \text{if} \quad \lambda > \lambda_d.$$

$$(\log 1000 \sim 7; \log t = 1000 \Rightarrow t \sim 10^{434}.)$$

## Exclusion Processes

(Spitzer, 1970)

From a 2009 paper by N. Bogoliubov (Steklov Mathematical Institute, St. Petersburg, Russia):

"The totally asymmetric simple exclusion process (TASEP) ... is connected with the 'crystalline limit' of the XXZ R-matrix. It is one of the most studied models of low dimensional non-equilibrium physics."

In one dimension, the state of the system is:

$$\cdots\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ \cdots$$

At random (exponential) times, particles try to move to the right with probability $p$ and to the left with probability $q = 1 - p$. Moves to occupied sites are not allowed.

Suppose $p = 1$ and the initial configuration is

$$\cdots\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ \cdots$$

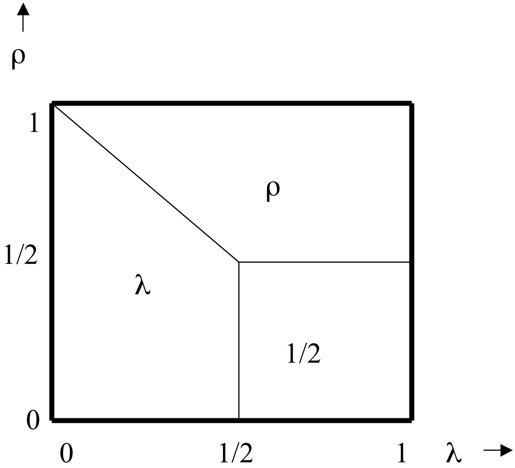
Question: What is the limiting distribution as $t \to \infty$?
Answer: Independent fair coin tossing, i.e.,

$$\lim_{t\to\infty} P(1 \text{ at time } t) = \frac{1}{2}, \quad \lim_{t\to\infty} P(11 \text{ at time } t) = \frac{1}{4}, \ldots$$

Question: What if $p$ is general, and the initial distribution is

$$\cdots\ \lambda\ \lambda\ \lambda\ \lambda\ \lambda\ \rho\ \rho\ \rho\ \rho\ \rho\ \cdots?$$

Answer: If $p = \frac{1}{2}$, the limiting distribution is independent coin tossing with parameter $\frac{\lambda+\rho}{2}$. If $p > \frac{1}{2}$, the parameter is


ρ
1
ρ
1/2
λ
1/2
0
0
1/2
1
λ

## The PDE Connection

Rescale space and time, pretending that the states of sites are independent. The result, letting

$$u(t,x) = \text{ the probability that site } x \text{ is occupied at time } t:$$

1. If $p = \frac{1}{2}$, the heat equation,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}.$$

2. If $p \neq \frac{1}{2}$, Burger's equation,

$$\frac{\partial u}{\partial t} + (p-q)\frac{\partial}{\partial x}\big[u(1-u)\big] = 0.$$

## Rates of Convergence for Symmetric Markov Chains

(Key words: Gibbs sampler; Markov Chain Monte Carlo — $(MC)^2$)

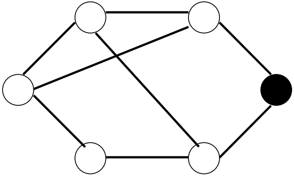
Consider a finite graph, with nonnegative weights $c_e$ attached to the edges.

Labels are places on the vertices.

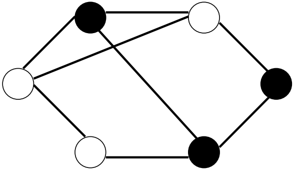
At exponential times of rate $c_e$, interchange the labels at the vertices joined by $e$.

Depending on the nature of the labels, various Markov chains can be defined:

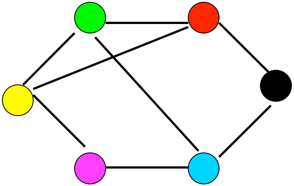
One particle MC



Symmetric exclusion



MC on permutations



For the Markov chain on permutations, think in terms of

## **Card Shuffling**

| **Vertices = positions in a deck** | **labels = cards** |
|---|---|
| 1 | Three of Clubs |
| 2 | Two of Diamonds |
| ⋮ | ⋮ |
| i | Five of Diamonds |
| ⋮ | ⋮ |
| j | Ten of Spades |
| ⋮ | ⋮ |
| 52 | Jack of Diamonds |

At rate $c_{i,j}$, interchange **Five of Diamonds** and **Ten of Spades**.

Note: $52! \sim 10^{68} >>>>> 52$.

For a finite state Markov chain $X(t)$, the transition probabilities are

$$p_t(i,j) = P(X(t) = j \mid X(0) = i).$$

If $P(t)$ is the matrix with entries $p_t(i,j)$, then

$$P(t) = e^{Qt}.$$

The smallest positive eigenvalue of $\lambda$ of $-Q$ determines the rate of convergence to the equilibrium $\pi$:

$$p_t(i,j) \sim \pi(j) + a(i,j)e^{-\lambda t}, \quad t \uparrow \infty.$$

Let $\lambda$ be this eigenvalue for the one particle process, and $\lambda^*$ be this eigenvalue for the process on permutations. Then

$$\lambda \geq \lambda^*.$$

**Aldous' Conjecture** (1992): $\lambda^* = \lambda$.

**Why guess this?**

1. True for $c_e \equiv 1$ on complete graph – Diaconis and Shahshahani (1981).

2. True for $c_e \equiv 1$ on star graphs – Flatto, Odlyzko and Wales (1985).

**Why do we care?**

1. It is MUCH easier to compute the eigenvalues of an $n \times n$ matrix than of an $n! \times n!$ matrix.

2. The main eigenvalue for the symmetric exclusion process lies between $\lambda$ and $\lambda^*$, so it would follow that it agrees with the common value of $\lambda$ and $\lambda^*$.

**Recent supporting results:**

1. True for general $c_e$ on trees – Handjani and Jungreis (1996).

2. Other related results by Koma and Nachtergele (1997), Morris (2008), Starr and Conomos (2008), Cesi (2009), and Dieker (2009).

**Theorem.** (Caputo, Liggett and Richthammer, 2009)

On a general finite graph with arbitrary rates,

$$\lambda^* = \lambda.$$