Stability on $\{0, 1, 2, \ldots\}^S$: birth-death chains and particle systems

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Abstract. A strong negative dependence property for measures on $\{0,1\}^n$ – *stability* – was recently developed in [5], by considering the zero set of the probability generating function. We extend this property to the more general setting of reaction-diffusion processes and collections of independent Markov chains. In one dimension the generalized stability property is now independently interesting, and we characterize the birth-death chains preserving it.

Mathematics Subject Classification (2000). Primary 60K35; Secondary 33C45, 60G50, 60J80.

Keywords. Stable polynomials, birth-death chain, negative association.

1. Introduction

In statistical physics a fundamental object of concern is the partition function, with its zeros having special relevance. For example, by introducing the effect of an external field, the partition function becomes a polynomial in the external field variable. As exemplified by the Lee-Yang circle theorem in the case of the Ising model [12], the general location of partition function zeros can indicate possible phase transitions.

A related object in probability is the *probability generating function.* However, the locations of its zeros were little studied before the recent work of Borcea, Brändén, and Liggett. In [5], a strong negative dependence theory for measures on $\{0,1\}^n$ was obtained; in particular, it was shown that if the generating function (in n variables) has no zeros with all imaginary parts positive, then the measure is negatively correlated in a variety of senses: negative association, ultra-log-concave rank sequence, Rayleigh property, and others.

The classification of linear transformations preserving the set of multivariate polynomials that are non-vanishing in circular regions was recently resolved in

A. Vandenberg-Rodes was partially support by NSF grants DMS-0707226 and NSF-0949250.

[2], with the investigation providing a general account of such polynomials and unifying several Lee-Yang-type theorems [3].

Our results are as follows: Using this framework we will generalize the negative dependence result in [5] to measures on $\{0,1,2,\dots\}^S$ – S countable – with application to independent Markov chains and reaction-diffusion processes. The one-coordinate case is also independently interesting; more specifically, the probability measures under consideration can be decomposed into a sum of independent Bernoulli and Poisson random variables.

Call such measures on $\{0,1,2,\dots\}$ *t-stable*. (The formal definition is given in Section 3.) In the last section we characterize the birth-and-death chains preserving this class of measures:

Theorem 1.1. *The birth-death chain $\{X_t; t \geq 0\}$ preserves the class of t-stable measures if and only if the birth rates are constant and the death rates satisfy $\delta_k = d_1 k + d_2 k^2$ for some constants d_1, d_2.*

One example is the pure death chain with rates $\delta_k = k(k-1)/2$, which expresses the number of ancestral geneologies in Kingman's coalescent – a well-studied model in mathematical biology [11, 10, 21]. In particular, by taking the initial number of particles to infinity, we obtain that the number of ancestors at any fixed time has the distribution of a sum of independent Bernoulli and Poisson random variables.

2. Stability and Negative Association

We first review the relationship – established in [5] – between negative association and the zero set of generating functions for measures on $\{0,1\}^n$.

Definition 2.1. A polynomial $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1,\dots,x_n]$ is called *stable* if $f \neq 0$ on the set

$$\mathbb{H}^n = \{(x_1,\dots,x_n) \in \mathbb{C}^n : \mathrm{Im}(x_j) > 0 \ \forall j\}.$$

Let $\mathfrak{G}[\mathbf{x}]$ be the set of all stable polynomials in the variable $\mathbf{x}$.

If f has only real coefficients, it also called *real* stable. The corresponding set of real stable polynomials is denoted $\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]$.

Note that a univariate real stable polynomial can only have real zeros.

One key fact from complex analysis is the (multivariate) Hurwitz's theorem on zeros of analytic functions: (see footnote 3 in [7])

Theorem 2.2. *Let Ω be a connected open subset of $\mathbb{C}^n$. Suppose the analytic functions $\{f_k\}$ converge uniformly on compact subsets of Ω (normal convergence in the vocabulary of complex analysis). If each f_k has no zeros in Ω then their limit f is either identically zero, or has no zeros in Ω. In particular, a normal limit of stable polynomials with bounded degree is either stable or* 0.

For μ a probability measure on $\{0,1\}^n$, let

$$f_\mu(x_1,\dots,x_n) = \sum_{i_1,\dots,i_n=0}^{1} \mu(i_1,\dots,i_n)x_1^{i_1}\cdots x_n^{i_n} = \mathbb{E}^\mu x_1^{\eta(1)}\cdots x_n^{\eta(n)}. \tag{2.1}$$

The last expression is just compact notation for the middle sum – the $\eta(i)$ are the coordinate variables for μ. f_μ is known as the *probability generating function* for μ. With this identification between measures and polynomials, we will freely abuse notation by referring to measures with stable generating functions as *stable measures* (such measures are also termed *Strongly Rayleigh* [5], by their connection with the Rayleigh property).

The concept of stability easily generalizes to countably many coordinates – a measure μ on $\{0,1\}^S$ is stable if every projection of μ onto finite subsets of coordinates is stable.

While the definition of stability is purely analytic, it implies two strong probabilistic conditions. Recall that a probability measure μ is negatively associated (NA) if, for all increasing continuous functions F, G depending on disjoint sets of coordinates,

$$\int FGd\mu \le \int Fd\mu \int Gd\mu.$$

The following was proved in [5]:

Theorem 2.3. *Suppose f_μ is stable. Then μ is NA.*

The second (and less difficult) probabilistic consequence of stability was given in [15, 22]:

Theorem 2.4. *Suppose μ is a measure on $\{0,1\}^S$ such that f_μ is stable. Then for any $T \subset S$,*

$$\sum_{i\in T}\eta(i) \overset{d}{=} \sum_{i\in T}\zeta_i,$$

where the ζ_i are independent Bernoulli variables.

3. Stable measures on $\{0,1,2,\dots\}^S$

Suppose μ is a measure on $\{0,1,2,\dots\}^n$. The generating function of μ is now the formal power series

$$f_\mu(x_1,\dots,x_n) = \sum_{i_1,\dots,i_n=0}^{\infty} \mu(i_1,\dots,i_n)x_1^{i_1}\cdots x_n^{i_n}. \tag{3.1}$$

If μ has finite support, then f_μ is a polynomial. In this case, let N be the maximum degree of f_μ in any of the variables $x_1,\dots,x_n$. We will want to represent f_μ by a multi-affine polynomial. To do this, we recall the *k-th elementary symmetric polynomial in m variables*

$$e_0 = 1, \quad e_k(x_1, \dots, x_m) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{3.2}$$

Then for a univariate polynomial $f(x) = \sum_{k=0}^{N} a_k x^k$ we define its *N-th polarization* as

$$Pol_N f(x_1, \dots, x_N) := \sum_{k=0}^{N} \binom{N}{k}^{-1} a_k e_k(x_1, \dots, x_N).$$

The N-th polarization of a multivariate polynomial is then defined to be the composition of polarizations in each variable. By considering $x_1 = \cdots = x_N = x$, notice that if $Pol_N f \in \mathfrak{G}[\mathbf{x}]$ then $f \in \mathfrak{G}[x]$. The converse also holds:

Theorem 3.1. *(Grace-Walsh-Szegö). Suppose f has degree at most N. Then f is stable iff $Pol_N f$ is stable.*

Many proofs of this result and its equivalent forms exist; see the appendix of [3], or [19, chapter 5].

Definition 3.2. We say that a function $f(\mathbf{x})$ defined on $\mathbb{C}^n$ is *transcendental stable*, or *t-stable*, if there exist stable polynomials $\{f_m(\mathbf{x})\}$ such that $f_m \to f$ uniformly on all compact subsets of $\mathbb{C}^n$ (f can then be expressed as an absolutely convergent power series on $\mathbb{C}^n$). Let $\overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$ be the set of all t-stable functions. This is also known as the Laguerre-Polya class [13].

We will again abuse notation and say that a measure μ on $\mathbb{N}^n$ is *transcendental stable* (or *t-stable*) if its generating function lies in $\overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$. Similarly, if μ has finite support and its generating polynomial is stable then we say that μ is stable. Of course, a stable measure is automatically t-stable.

The papers by Borcea and Brändén [2, 4] characterized the linear transformations preserving stable polynomials by establishing a bijection between linear transformations preserving n-variable stability and t-stable powers series in $2n$ variables. We will not require their full result here; however, the following characterization of t-stable powers series – the technical cornerstone upon which the above bijection rests – will be most useful.

Recall the standard partial order on $\mathbb{N}^n$: $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $1 \le i \le n$. Also, for $\alpha \le \beta \in \mathbb{N}^n$ define

$$(\beta)_\alpha = \frac{\beta!}{(\beta - \alpha)!} \text{ if } \alpha \le \beta, \quad (\beta)_\alpha = 0 \text{ otherwise.} \tag{3.3}$$

Theorem 3.3 (Theorem 6.1 of [2]**).** *Let $f(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{x}^\alpha$ be a formal power series in $\mathbf{x}$ with coefficients in $\mathbb{R}$. Set $\beta_m = (m, m, \dots, m) \in \mathbb{N}^n$. Then $f(\mathbf{x}) \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$ if and only if*

$$f_m(\mathbf{x}) := \sum_{\alpha \le \beta_m} (\beta_m)_\alpha c_\alpha \left(\frac{\mathbf{x}}{m}\right)^\alpha \in \mathfrak{G}_{\mathbb{R}}[\mathbf{x}] \cup \{0\}, \tag{3.4}$$

for all $m \in \mathbb{N}$. In this case, the polynomials $f_m(\mathbf{x}) \to f(\mathbf{x})$ uniformly on compact sets.

This classification also has the following immediate consequence:

Corollary 3.4. *The class $\overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$ is closed under convergence of coefficients. In particular, the set of t-stable probability measures on $\mathbb{N}^n$ is closed under weak convergence.*

Proof. Suppose that for each n,

$$f^{(n)}(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha^{(n)} \mathbf{x}^\alpha \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]},$$

with $c_\alpha^{(n)} \to c_\alpha$ for each α. Then for each m the stable polynomials $f_m^{(n)}(\mathbf{x})$, defined in (3.4) above, converge normally to the polynomial f_m likewise obtained from f. Hurwitz's Theorem implies that each f_m is stable, and applying Theorem 3.3 again we conclude that $f \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$. □

By the following proposition, we can say that a measure on $\{0, 1, 2, \dots\}^S$ is t-stable if every projection onto a finite subset of coordinates is a t-stable measure.

Proposition 3.5. *The class of t-stable measures is closed under projections onto subsets of coordinates.*

Proof. It suffices to consider projections of n coordinates onto $n-1$ coordinates. Suppose that μ is a t-stable measure on $\mathbb{N}^n$. If $f(x_1, \dots, x_n)$ is its generating function, notice that the generating function of the projection of μ onto $\mathbb{N}^{n-1}$ is $f(x_1, \dots, x_{n-1}, 1)$. By Theorem 3.3, it suffices to show that the approximating polynomials $f_m(x_1, \dots, x_{n-1}, 1)$ are stable. But this follows by considering the (complex) stable polynomials $f_m(x_1, \dots, x_{n-1}, 1 + i/k)$ and applying Hurwitz's theorem as $k \to \infty$. □

We can now give an extension of Theorem 2.3.

Theorem 3.6. *Suppose μ is a t-stable probability measure on $\{0, 1, 2, \dots\}^S$. Then μ is NA.*

Proof. By the previous proposition and a limiting argument it is sufficient to show the result for measures on $\mathbb{N}^n$. Let $f(\mathbf{x})$ be the generating function of μ. By definition, $f \in \overline{\mathfrak{G}_{\mathbb{R}}[\mathbf{x}]}$. Let $\{f_N(\mathbf{x})\}$ be the stable polynomials converging to f as in Theorem 3.3; we can also normalize them so that $f_N(1) = 1$. Let μ_N be the respective probability measures on $\{0, 1, 2, \dots, N\}^n$. Hence by the Grace-Walsh-Szegö (GWS) theorem, $Pol_N f_N$ is the generating function for a stable measure $\tilde{\mu}_N$ on $\{0, 1\}^{nN}$. Let

$$\{\zeta_{ij}; 1 \le i \le n, 0 \le j \le N\}$$

be the coordinates of $\tilde{\mu}_N$, such that $\eta_i = \sum_j \zeta_{ij}$ is the i-th coordinate of μ_N. Hence for bounded increasing functions F and G on $\{0, 1, 2, \dots\}^n$, depending on disjoint sets of coordinates, we have

$$\begin{aligned}
&\mathbb{E}^{\mu_N}[F(\eta_1,\ldots,\eta_n)G(\eta_1,\ldots,\eta_n)]\\
&= \mathbb{E}^{\mu_N}\left[F\Big(\sum_j \zeta_{1j},\ldots,\sum_j \zeta_{nj}\Big)G\Big(\sum_j \zeta_{1j},\ldots,\sum_j \zeta_{nj}\Big)\right]\\
&\le \mathbb{E}^{\mu_N}\left[F\Big(\sum_j \zeta_{1j},\ldots,\sum_j \zeta_{nj}\Big)\mathbb{E}^{\mu_N}\left[G\Big(\sum_j \zeta_{1j},\ldots,\sum_j \zeta_{nj}\Big)\right.\right.\\
&= \mathbb{E}^{\mu_N}F(\eta_1,\ldots,\eta_n)\mathbb{E}^{\mu_N}G(\eta_1,\ldots,\eta_n)
\end{aligned}$$

The inequality above follows because $F(x_{11}+\cdots+x_{1N},\ldots,x_{n1}+\cdots+x_{nN})$ is an increasing function in the nN variables (similarly with G), and the ζ_{ij} are all negatively associated by Theorem 2.3. The normal convergence of $f_N \to f$ implies the weak convergence $\mu_N \to \mu$, concluding the proof. □

We can also characterize all t-stable measures on one coordinate.

Proposition 3.7. *A probability measure on* $\{0,1,2,\ldots\}$ *is transcendental stable if and only if it has the same distribution as a (possibly infinite) sum of independent Bernoulli random variables and a Poisson random variable.*

Proof. Suppose f is a t-stable generating function for a non-negative, integer valued random variable. By Theorem 3.3, f is a normal limit of univariate polynomials with all zeros on the negative real axis. An appeal to the classical theory of entire functions (e.g. [13, VIII, Theorem 1]) indicates that f can be expressed as the following infinite product:

$$f(x) = Cx^q e^{\sigma x}\prod_{k=1}^{\infty}\left[1-\frac{x}{a_k}\right],$$

for some $q\in\mathbb{N}$, $\sigma\ge 0$, $a_k<0$, and $\sum |a_k|^{-1}<\infty$. A little rearrangement – using the fact that $f(1)=1$ – gives the following alternative expression:

$$f(x) = x^q e^{\sigma(x-1)}\prod_{k=1}^{\infty}[(1-p_k)+xp_k],$$

where $p_k = 1/(1-a_k)$. This we recognize as the generating function for the sum of a non-negative constant q, independent Poisson(σ) and Bernoulli(p_k) random variables. Conversely, any generating function of this form with $\sum p_k < \infty$ is automatically t-stable, as e^x is the normal limit of the polynomials $(1+x/n)^n$. □

By projecting onto finite subsets of coordinates (taking limits if need be) and setting all variables in the resulting generating function to be equal, we obtain the following extension of Theorem 2.4:

Corollary 3.8. *Suppose* μ *is a t-stable measure on* $\mathbb{N}^S$*. Then for any* $T\subset S$ *the number of particles located in* T *– according to* μ *– has the distribution of a sum of independent Bernoulli and Poisson random variables.*

3.1. Markov processes and stability

Suppose $\{\eta_t; t \geq 0\}$ is a Markov process on $\mathbb{N}^n$. We define the associated linear operator T_t on power series with bounded coefficients by letting $T_t(\mathbf{x}^\alpha)$ be the generating function of $\{\eta_t | \eta_0 = \alpha\}$ for each $\alpha \in \mathbb{N}^n$, and extending by linearity. This is well-defined because $\sum_{k \geq 0} P(\eta_t = k) = 1$.

Definition 3.9 (Preservation of stability). We say that a Markov process $\{\eta_t; t \geq 0\}$ on $\mathbb{N}^n$ *preserves stability* if for any stable initial distribution, the distribution at any later time is t-stable. That is, the associated linear operator T_t maps the set of stable polynomials with non-negative coefficients into the set of t-stable power series.

The process η_t *preserves t-stability* if for any t-stable initial distribution, the distribution at a later time is again t-stable. That is, T_t maps the set of t-stable power series with non-negative coefficients into itself.

In fact, these two definitions are equivalent.

Proposition 3.10. *A Markov process preserves t-stability if and only if it preserves stability.*

Proof. Only one direction needs proof. Assume the process preserves stability. Let T_t be the associated linear operator, and $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ be the generating function of a t-stable distribution; hence $c_\alpha \geq 0$ for all α and $\sum_\alpha c_\alpha = 1$. By Theorem 3.3 there are stable polynomials $f_n = \sum_\alpha c_\alpha^{(n)} \mathbf{x}^\alpha$ with $c_\alpha^{(n)} \to c_\alpha$, all $c_\alpha^{(n)} \geq 0$, and with $\sum_\alpha c_\alpha^{(n)} \leq 1$. Suppose that

$$T_t(\mathbf{x}^\alpha) = \sum_\beta d_{\alpha,\beta} \mathbf{x}^\beta.$$

Since probability is conserved, $\sum_\beta d_{\alpha,\beta} = 1$, and hence by dominated convergence,

$$\sum_\alpha c_\alpha^{(n)} d_{\alpha,\beta} \overset{n\to\infty}{\longrightarrow} \sum_\alpha c_\alpha d_{\alpha,\beta}.$$

In other words, the coefficients of $T_t f_n$ (which is t-stable by assumption), converge to the coefficients of $T_t f$. $T_t f$ is then t-stable by Corollary 3.4. $\square$

We now give a couple examples.

3.2. Independent Markov chains

Suppose $\{X_t(1), X_t(2), \ldots\}$ is a collection of independent Markov chains on S with identical jump rates. Set

$$\eta_t(x) = \sum_{i \geq 1} 1_{\{X_t(i) = x\}},$$

so that the resulting process is a collection of particles on S jumping independently with the same rates. This is well defined as long as $\eta_t(x) < \infty$ for all $x \in S,\ t \geq 0$ – one possibility is to restrict initial configurations to the space E_0 defined below for reaction-diffusion processes.

Proposition 3.11. *The process $\{\eta_t; t \geq 0\}$ preserves t-stability. Hence, assuming that the initial distribution is t-stable, the distribution at any time is negatively associated by Theorem* 3.6.

Proof. Let μ_t be the distribution of η_t, with μ_0 t-stable. We need to show that for each finite $T \subset S$, the projection $\mu_t|_T$ is t-stable. Taking finite $T \subset S_1 \subset S_2 \subset \cdots$ with each S_n finite and $S_n \nearrow S$, we can approximate $\mu_t|_T$ by the sequence $\mu_t^{(n)}|_T$, with each $\mu_t^{(n)}$ the distribution of the independent Markov chain process on S_n given initial distribution $\mu_0|_{S_n}$ and jumps restricted to staying inside S_n. Hence by Corollary 3.4 we can assume finite S.

Suppose now that $S = [n]$, and only jumps from site 1 to site 2 are allowed. In this case, assuming a jump rate $q(1,2)$, each particle at x independently has probability $p := 1 - e^{-tq(1,2)}$ of moving to y. Hence the associated linear operator T_t takes

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mapsto (px_2 + (1-p)x_1)^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

that is,

$$T_t f(x_1, \ldots, x_n) = f(px_2 + (1-p)x_1, x_2, \ldots, x_n),$$

which preserves the class of stable polynomials. By permuting variables, this argument also holds for general $i, j \in [n]$.

Recalling the Banach space $C_0(\mathbb{N}^n)$ of functions that vanish at infinity, consider the strongly continuous contraction semi-groups $S^{i,j}(t)$ on $C_0(\mathbb{N}^n)$ defined by

$$S^{i,j}(t) f(\eta) = \mathbb{E}^\eta f(\eta_t^{i,j}),$$

where $\{\eta_t^{i,j}; t \geq 0\}$ is the (Feller) process of independent Markov chains which only allow jumps from site i to site j. Then for a t-stable intial distribution μ, we just showed that $\mu S^{i,j}(t)$ – the distribution of $\eta_t^{i,j}$ assuming initial distribution μ – is again t-stable. By Trotter's product theorem [9, p. 33], the process allowing the jumps $\{i \mapsto j\}, \{k \mapsto l\}$ has semigroup

$$S(t) = \lim_{n \to \infty} \left[S^{i,j}\Big(\frac{t}{n}\Big) S^{k,l}\Big(\frac{t}{n}\Big) \right]^n.$$

Hence by Corollary 3.4, $\mu S(t)$ is again t-stable. Including all the possible jumps one-by-one, we conclude that the whole process $\{\eta_t; t \geq 0\}$ on $\mathbb{N}^n$ preserves t-stability. □

Remark 3.12. Let $\mathcal{M}$ be the set of probability measures on $\mathbb{N}^S$ described by random configurations η with coordinates

$$\eta(k) = \sum_i 1_{\{Y_i = k\}},$$

where $Y_1, Y_2, \ldots$ are independent random variables with values in $S \cup \{\infty\}$. In [14] it was shown that $\mathcal{M}$ is preserved by the process of independent Markov chains, and that measures in $\mathcal{M}$ are NA. Proposition 3.11 is a generalization of this result, since it is easily checked that the class $\mathcal{M}$ is contained in the class of t-stable measures. Indeed, if $\mu \in \mathcal{M}$ and $S = [n]$, then μ has a generating function of the form

$$\begin{aligned} &\mathbb{E} x_1^{\sum_i 1_{\{Y_i = 1\}}} \cdots x_n^{\sum_i 1_{\{Y_i = n\}}} \\ &= \prod_i [P(Y_i = \infty) + P(Y_i = 1)x_1 + \cdots + P(Y_i = n)x_n], \end{aligned}$$

by of the independence of the $Y_i's$. Furthermore, (non-constant) product measures for which each coordinate is a sum of independent Bernoulli and Poisson measures are t-stable, but are not contained in the class $\mathcal{M}$.

3.3. Reaction-diffusion processes

In addition to having the motion of particles following independent Markov chains, we can also allow particles to undergo a reaction at each site. Let $p(i, j)$ be transition probabilities for a Markov chain on S. Given a state $\eta \in \mathbb{N}^S$, we consider the following evolution:

1. at rate $\beta^i_{\eta(i)}$ a particle is created at site i,
2. at rate $\delta^i_{\eta(i)}$ a particle at site i dies.
3. at rate $\eta(i)p(i, j)$ a particle at site i jumps to site j.

The most common example is the *polynomial model*, where the birth-death rates for each site are

$$\beta_k = \sum_{j=0}^{m-1} b_j x(x-1)\cdots(x-j+1), \quad \delta_k = \sum_{j=1}^{m} d_j x(x-1)\dots(x-j+1).$$

Reaction-diffusion processes originated as a model for chemical reactions [18], and subsequent work by probabilists has focused on the ergodic properties [8, 6, 1].

To construct the process, we require a strictly positive sequence k_i on the index set S, and a positive constant M such that

$$\sum_j p(i, j) k_j \le M k_i, \quad i \in S.$$

Furthermore, the birth rates must satisfy

$$\sum_i \beta^i_0 k_i < \infty.$$

Then we take the state space of the process to be

$$E_0 = \{\eta \in \mathbb{N}^S : \sum_i \eta(i) k_i < \infty\}.$$

See [6, chapter 13.2] for the details of the construction.

In a very simple case we have preservation of t-stability:

Proposition 3.13. *Suppose the reaction-diffusion process on $\mathbb{N}^S$ is well-constructed with $\beta^i_k = b^i$ and $\delta^i_k = d^i k$ – this is the polynomial model with $m = 1$ and site-varying reaction rates. Then the process preserves t-stability, and hence – assuming a t-stable initial configuration – its distribution at any time is negatively associated.*

Proof. The strategy here is the same as with Proposition 3.11. To reduce to a reaction-diffusion process on a finite number of sites, we approximate using the construction in [6, Theorem 13.8]. Furthermore, on the locally compact space $\mathbb{N}^n$, the reaction-diffusion process with at most constant birth and linear death rates is now a Feller process, as can be seen from [9, Theorem 3.1, Ch. 8]. Hence by Trotter's product formula and Corollary 3.4 we only need to show that the following processes preserve stability on $\mathbb{N}^n$:

1. constant birth rate b^i at a single site i.
2. linear death rates at a single site i ($\delta_k^i = d^i k$),
3. jumps from site i to site j at rate $\eta(i)p(i,j)$

For (1), we note that with a constant birth rate, at time t a Poisson($b^i t$) number of particles has been added to the system – i.e. the original generating function is multiplied by $e^{b^i t(x_i-1)}$, preserving t-stability.

(2) can be thought of as the process in which each particle at site i dies independently at rate d^i. Hence the associated linear transform is defined by

$$T_t(x_1^{\alpha_1}\cdots x_i^{\alpha_i}\cdots x_n^{\alpha_n}) = x_1^{\alpha_1}\cdots[1-e^{-d^i t}+e^{-d^i t}x_i]^{\alpha_i}\cdots x_n^{\alpha_n}.$$

As the affine transformation $x \mapsto ax+(1-a)$, $(a>0)$ maps the upper half plane onto itself, T_t preserves preserves the class of stable polynomials.

Finally, we note that (3) was already seen to preserve stability from the proof for independent Markov chains. □

Other reaction-diffusion processes do not preserve stability in general. On one coordinate, a reaction-diffusion process is just a birth-death chain, so by Theorem 1.1 the only possible generalization would be to quadratic death rates. In this case, unfortunately, the associated linear transformation T_t will *not* preserve stable polynomials with positive roots; indeed, assuming death rates $\delta_k = k(k-1)$, quadratic polynomials with a double root inside the interval $(0,1)$ will not be mapped to stable polynomials under T_t. A much more complicated example – which we shall not reproduce here – shows that quadratic death rates on multiple sites does not preserve the class of stable probability measures.

4. Birth-Death Chains

Our goal in this section is to prove Theorem 1.1. As just noted above, in the case of quadratic death rates the associated linear transformation does not preserve all polynomials with real zeros, and hence we cannot rely on the classification theory in [2]. From the probabilistic point of view, it would be useful to have a similar theory for linear transformations on polynomials with positive coefficients; however, in what follows we will make do with several perturbation arguments – the main idea being that a polynomial's roots move continuously under changes to its coefficients.

Let $\{X_t; t \geq 0\}$ be a non-explosive birth and death chain on $\{0, 1, 2, ...\}$ with $Q-$matrix given by

$$q(k, k+1) = \beta_k, \quad q(k, k-1) = \delta_k, \quad \text{and} \quad q(k, k) = -\beta_k - \delta_k,$$

with $\delta_0 = 0$. See Chapter 2 of [16] for the relevant definitions. The generating function at time t is given by

$$\phi(t, z) = \sum_{k=0}^{\infty} P(X_t = k) z^k.$$

By Theorem 2.14 of [16], the transition probabilities $p_t(j, k) = P^j(X_t = k)$ are continuously differentiable in t and satisfy the Kolmogorov backward equations

$$\frac{d}{dt} p_t(j, k) = \beta_j p_t(j+1, k) + \delta_j p_t(j-1, k) - (\beta_j + \delta_j) p_t(j, k).$$

By Theorem 2.13 of [16], $|p_t(j, k) - p_s(j, k)| \leq 1 - p_{|t-s|}(j, j)$. It follows that

$$\frac{\partial}{\partial t} \phi(t, z) = \sum_{k=0}^{\infty} \frac{d}{dt} P(X_t = k) z^k$$

for $|z| < 1$, provided that $X(0)$ is bounded. Iterating this argument, one sees that $\phi(t, x)$ is C^2 on $[0, \infty) \times \{z : |z| < 1\}$ if $X(0)$ is bounded.

Proposition 4.1. *Suppose the birth-death process $\{X_t; t \geq 0\}$ preserves stability. Then there exist constants b_0, b_1, b_2, d_1, d_2 so that*

$$\beta_k = b_0 + b_1 k + b_2 k^2 \quad \textit{and} \quad \delta_i = d_1 k + d_2 k^2.$$

Proof. Suppose

$$\phi(0, z) = c \prod_{k=1}^{n} (z - z_k),$$

where $c > 0$, $n \geq 3$, and $-1 < z_1, ..., z_n < 0$. By assumption, the generating function $\phi(t, z)$ of X_t has only real roots for $t > 0$. If $z_1 = z_2 = w$ is a root of $\phi(0, z)$ of multiplicity exactly two, and ϵ is small enough that $|z_k - w| > \epsilon$ for $k \geq 3$, Rouché's Theorem implies that for sufficiently small $t > 0$, $\phi(t, z)$ has exactly two roots in the disk $\{z : |z - w| < \epsilon\}$. Therefore, for small $t > 0$, there exist real $z(t)$ so that $\phi(t, z(t)) = 0$ and $\lim_{t \downarrow 0} z(t) = w$. By Taylor's Theorem, there exist $s(t) \in [0, t]$ and $y(t)$ between $z(t)$ and w so that

$$\begin{aligned} 0 =& \phi(t, z(t)) = t \frac{\partial \phi}{\partial t}(0, w) + \frac{1}{2} t^2 \frac{\partial^2 \phi}{\partial t^2}(s(t), y(t)) \\ &+ t(z(t) - w) \frac{\partial^2 \phi}{\partial t \partial z}(s(t), y(t)) + \frac{1}{2}(z(t) - w)^2 \frac{\partial^2 \phi}{\partial z^2}(s(t), y(t)). \end{aligned}$$

Dividing by t and letting $t \downarrow 0$ leads to

$$2 \frac{\partial \phi}{\partial t}(0, w) \Big/ \frac{\partial^2 \phi}{\partial z^2}(0, w) = -\lim_{t \downarrow 0} \frac{(z(t) - w)^2}{t} \leq 0.$$

Noting that

$$\frac{\partial^2\phi}{\partial z^2}(0,w) = 2c\prod_{k=3}^{n}(z-z_k),$$

we see that

$$\frac{\partial\phi}{\partial t}(0,w) \tag{4.1}$$

changes sign when z_3 crosses w, and hence is zero when $z_3 = w$.

To exploit this fact, we need to compute (4.1). Recall the k^{th} elementary symmetric polynomials $e_k(x_1,\dots,x_n)$ defined in (3.2). If μ is the distribution of X_0 and $X_0 \le n$, then

$$\phi(0,z) = \sum_{k=0}^{n}\mu(k)z^k = c\prod_{k=1}^{n}(z-z_k) = c\sum_{k=0}^{n}(-1)^k e_k(z_1,...,z_n)z^{n-k},$$

so

$$\mu(k) = c(-1)^{n-k}e_{n-k}(z_1,...,z_n).$$

Therefore for $|z|<1$,

$$\frac{\partial\phi}{\partial t}(0,z) = \sum_{l=0}^{\infty}\sum_{k=0}^{n}\mu(k)q(k,l)z^l = c(1-z)\sum_{k=0}^{n}(-1)^{n-k}e_{n-k}(z_1,...,z_n)[\delta_k z^{k-1} - \beta_k z^k].$$

It follows that the expression on the right is zero if $z = z_1 = z_2 = z_3 = w$ for any values of $w, z_4, ..., z_n \in (-1,0)$. In this case,

$$e_k(z_1,...,z_n) = \sum_i w^i\binom{3}{i}e_{k-i}(z_4,...,z_n),$$

where i ranges from $\max(0, k+3-n)$ to $\min(k,3)$, so

$$\sum_{k=0}^{n}\sum_i(-1)^{n-k}\binom{3}{i}e_{n-k-i}(z_4,...,z_n)[\delta_k w^{k+i-1} - \beta_k w^{k+i}] \equiv 0.$$

Interchanging the order of summation and letting $k \mapsto k-i$, we see that the coefficient of each of the e_{n-k}'s is zero:

$$\sum_i\binom{3}{i}(-1)^i[\delta_{k-i} - \beta_{k-i}w] = 0,$$

or equivalently $\delta_k - 3\delta_{k+1} + 3\delta_{k+2} - \delta_{k+3} = 0$ and $\beta_k - 3\beta_{k+1} + 3\beta_{k+2} - \beta_{k+3} = 0$, so that the birth rates β_k and death rates δ_k are quadratic functions of k. □

With the next proposition we resolve the "only if" part of Theorem 1.1.

Proposition 4.2. *The birth-death chain preserves stability only if the birth rate is constant.*

Proof. Assuming that the chain preserves stability, we will show the birth rates β_k satisfy $\beta_k \geq \beta_{k+1}$ for each k, so by Proposition 4.1 β_k is constant.

By iterating the Kolmogorov backward equations, one can obtain the following approximations for small $t > 0$:

$$p_t(k,k+1) = t(\beta_k + o(1)), \quad p_t(k,k+2) = \frac{t^2}{2}(\beta_k\beta_{k+1} + o(1)).$$

Similarly,

$$p_t(k,k-j) = O(1)t^j,$$

where $O(1)$ denotes a uniformly bounded quantity, and $o(1) \to 0$, as $t \to 0$.

Suppose that we start the chain with k particles; the initial distribution has generating function $f(x) = x^k$. We also can assume that $\beta_k, \beta_{k+1} > 0$. Then the generating function for small $t > 0$ will be:

$$f_t(x) = \cdots + (1+o(1))x^k + (\beta_k + o(1))tx^{k+1} + (\beta_k\beta_{k+1} + o(1))\frac{t^2}{2}x^{k+2} + \cdots$$

Since $f_t(x)$ is t-stable, by Theorem 3.3 the following polynomial has all real, negative roots:

$$f_{t,k+2}(x) = \cdots + (1+o(1))x^k + \frac{2(\beta_k + o(1))}{k+2}tx^{k+1} + \frac{\beta_k\beta_{k+1} + o(1)}{(k+2)^2}t^2x^{k+2}.$$

As the hidden coefficients are $o(1)$, Rouché's Theorem implies that k roots of $f_{t,k+2}$ are also $o(1)$. Thus the remaining two roots a, b satisfy

$$a + b = \frac{2(k+2) + o(1)}{\beta_{k+1}t}$$
$$ab = \frac{(k+2)^2 + o(1)}{\beta_k\beta_{k+1}t^2}.$$

Solving for real a, b implies that the discriminant

$$4t^{-2}(k+2)^2[\beta_{k+1}^{-2} - (\beta_k\beta_{k+1})^{-1} + o(1)] \geq 0, \quad \text{for small } t > 0.$$

Taking $t \to 0$, we conclude that $\beta_k \geq \beta_{k+1}$. $\square$

We now concentrate on the "if" part of Theorem 1.1. For what follows, recall the notation in (3.3).

Lemma 4.3. *Suppose $p(k)$ is a polynomial of degree at most r. then*

$$\sum_{k=l}^{n}(-1)^k\binom{n}{k}\binom{k}{l}p(k) = 0 \text{ for all } l < n - r.$$

Proof. First use the easy identity

$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}.$$

Now for $0 \le q \le r$,

$$\begin{aligned}&\sum_{k=l}^{n}(-1)^k\binom{n}{l}\binom{n-l}{k-l}(k-l)_q\\ =&\sum_{k=l+q}^{n}(-1)^k\binom{n}{l}\binom{n-l-q}{k-l-q}(n-l)_{q-1}\\ =&\binom{n}{l}(n-l)_{q-1}(1-1)^{n-l-q}=0.\end{aligned}$$

As any polynomial of degree at most r can be written as a linear combination of the polynomials in k:

$$\{1,(k-l)_1,\dots,(k-l)_r\},$$

this concludes the proof. □

Proposition 4.4. *The birth-death chain with quadratic death rates $\beta_k = 0$, $\delta_k = k(k-1)$ preserves stability.*

Proof. Let $\phi(t,z)$ be the generating function of the chain at time t. Setting $\tau = \inf\{t \ge 0;\ \phi(t,z)$ is not stable$\}$, by Hurwitz's Theorem $\phi(\tau,z)$ is stable. Hence by time-homogeneity of the birth-death chain, it suffices to prove that for any stable initial distribution there exists an $\epsilon > 0$ such that $\phi(t,z)$ is stable for all $0 < t < \epsilon$. Suppose first that

$$\phi(0,z) = c(z-w)^n, \quad w<0, \quad \phi(0,1)=1. \tag{4.2}$$

We will show that for all small enough $t > 0$, $\phi(t,z)$ has n real zeros at a distance of order $t^{1/2}$. Indeed, by Taylor expanding $\phi(t,z)$ in t we will see that

$$\frac{d^k}{ds^k}\bigg|_{s=0}\phi(s,w+\alpha t^{\frac{1}{2}}) = O(t^{\frac{n}{2}-k}), \text{ and} \tag{4.3}$$

$$\phi(t,w+\alpha t^{1/2}) = ct^{n/2}p(\alpha)+o(t^{n/2}), \tag{4.4}$$

where $p(\alpha)$ is essentially the n'th Hermite polynomial, which has n distinct real zeros [20].

By (4.4), we see that for small enough t, $\phi(t,z)$ changes sign n times near w, and hence has n real zeros.

The general case then follows easily. For example, if

$$\phi(0,z) = c(z-w_1)^{n_1}(z-w_2)^{n_2} = \phi_1(0,z)\phi_2(0,z),$$

then

$$\frac{d^k}{ds^k}\bigg|_{s=0}\phi(s,w_1+\alpha t^{\frac{1}{2}}) = \sum_{j=0}^{k}\frac{d^j}{ds^j}\binom{k}{j}\bigg|_{s=0}\phi_1(s,w_1+\alpha t^{\frac{1}{2}})\frac{d^{k-j}}{ds^{k-j}}\bigg|_{s=0}\phi_2(s,w_1+\alpha t^{\frac{1}{2}}).$$

By (4.3), the terms with $j \ne k$ contribute $o(t^{n_1/2})$ to the Taylor expansion around $w_1+\alpha t^{1/2}$, and can be ignored. The remaining terms thus give the same expression for $\phi(t, w_1 + \alpha t^{1/2})$ as in (4.4), and so again for small times t there are n_1 real zeros nearby w_1. Similarly, there are n_2 real zeros near w_2 also, so stability is preserved.

We now show (4.3) and (4.4) for $\phi(0, z)$ of the form (4.2). Let μ_t be the distribution at time t (and hence with $\phi(t, z)$ as its generating function).

Our first step is to compute all the derivatives

$$\frac{d^m}{dt^m}\phi(t,z)\bigg|_{t=0} = \sum_{k=0}^{n} \frac{d^m}{dt^m}\mu_t(k)\bigg|_{t=0} z^k.$$

Recall the notation in (3.3). By repeated use of the Kolmogorov backward equation and shifting the variable k by 1, we obtain:

$$\begin{aligned}
&\sum_{k=0}^{n}\left[(k+1)_2\frac{d^{m-1}}{dt^{m-1}}\mu_t(k+1)z^k - (k)_2\frac{d^{m-1}}{dt^{m-1}}\mu_t(k)z^k\right]\bigg|_{t=0} \\
&= \sum_{k=0}^{n}(k)_2\frac{d^{m-1}}{dt^{m-1}}\mu_t(k)(1-z)z^{k-1}\bigg|_{t=0} \\
&= \sum_{k=0}^{n}(1-z)(k)_2\left[(k-1)_2 z^{k-2} - (k)_2 z^{k-1}\right]\frac{d^{m-2}}{dt^{m-2}}\mu_t(k)\bigg|_{t=0} \\
&\vdots \\
&= \sum_{k=0}^{n}(1-z)\sum_{i_1=0}^{1}\cdots\sum_{i_{m-1}=0}^{1}(k)_2(k-i_1)_2\cdots(k-i_1-\cdots-i_{m-1})_2 \\
&\qquad\qquad \times z^{k-1-\sum_{l=1}^{m-1} i_l}(-1)^{\sum_{l=1}^{m-1}(1-i_l)}\mu(k).
\end{aligned}$$

This last expression follows by an induction argument.

Using the equivalence between binary strings $i_1 i_2 \dots i_{m-1}$ of length $m-1$ and subsets $A \subset [m-1]$, we can rewrite the last expression for the m-th derivative as

$$\sum_{k=0}^{n}(1-z)\sum_{j=0}^{m-1}\sum_{\substack{A\subset[m-1]\\|A|=j}} K_A^k z^{k-1-j}(-1)^{m-1-j}\mu(k), \tag{4.5}$$

where we let

$$K_A^k = (k)_2(k-|A\cap[1]|)_2(k-|A\cap[2]|)_2\cdots(k-|A\cap[m-1]|)_2.$$

By the definition of K_A^k, we see that when $|A| = j$, $K_A^k = 0$ for $k \le j$, hence in the above z is always raised to a non-negative integer power. Now we consider

$$z = w + \alpha t^{\frac{1}{2}},$$

and expand (4.5) with the binomial formula:

$$(1-z)\sum_{k=0}^{n}\sum_{j=0}^{m-1}\sum_{\substack{A\subset[m-1]\\|A|=j}} K_A^k \sum_{l=0}^{k-1-j}(-1)^{m-1-j}\binom{k-1-j}{l}(\alpha t^{\frac{1}{2}})^l w^{k-1-j-l}\mu(k).$$

Notice that

$$\mu(k)w^{k-1-j-l} = c(-1)^{n-k}\binom{n}{k}w^{n-j-l-1},$$

and reorder the summations to obtain

$$= c(-1)^{n+m-1}(1-z)\sum_{j=0}^{m-1}(-1)^j\sum_{l=0}^{n-1-j}(\alpha t^{\frac{1}{2}})^l w^{n-j-l-1} \times \sum_{k=l+j+1}^{n}(-1)^k\binom{n}{k}\binom{k-1-j}{l}\sum_{\substack{A\subset[m-1]\\|A|=j}}K_A^k. \tag{4.6}$$

When $|A| = j$, K_A^k contains the factor $k(k-1)\cdots(k-j) = (k)_{j+1}$. Hence we can rewrite

$$\sum_{\substack{A\subset[m-1]\\|A|=j}}K_A^k = (k)_{j+1}p(k),$$

where $p(k)$ is a polynomial of degree exactly $2m-j-1$. Thus

$$\begin{aligned}
&\sum_{k=l+j+1}^{n}(-1)^k\binom{n}{k}\binom{k-1-j}{l}\sum_{\substack{A\subset[m-1]\\|A|=j}}K_A^k\\
&= \sum_{k=l+j+1}^{n}(-1)^k\binom{n}{k}\binom{k-1-j}{l}(k)_{j+1}p(k)\\
&= \sum_{k=l}^{n}(-1)^k\binom{n}{k}\binom{k}{l}(k-l)_{j+1}p(k),
\end{aligned}$$

which by Lemma 4.3 is zero for $l < n-2m$.

We have shown that

$$\frac{d^m}{dt^m}\phi(t, w+\alpha t^{\frac{1}{2}})\bigg|_{t=0} = o(t^{\frac{n-2m}{2}}) + c(\alpha t^{\frac{1}{2}})^{n-2m}(-1)^{n+m-1}(1-z)\times \sum_{j=0}^{m-1}(-1)^j w^{2m-j-1}\sum_{k=n-2m+j+1}^{n}(-1)^k\binom{n}{k}\binom{k-1-j}{n-2m}\binom{m-1}{j}(k)_{j+1}p'(k), \tag{4.7}$$

where $p'(k)$ is a monic polynomial of degree exactly $2m-j-1$. Doing the same trick as above with Lemma 4.3, the sum over k can be written as

$$\binom{m-1}{j}\sum_{k=n-2m}^{n}(-1)^k\binom{n}{k}\binom{k}{n-2m}p''(k),$$

with $p''(k)$ a new monic polynomial of degree $2m$. By Lemma 4.3 again, we may choose any monic polynomial of degree $2m$, in particular, $p''(k) = (k-n+2m)_{2m}$.

Then all terms in the sum cancel save for $k = n$. After much simplification we can rewrite (4.7) as

$$\frac{d^m}{dt^m}\phi(t, w + \alpha t^{\frac{1}{2}})\bigg|_{t=0} = c(\alpha t^{\frac{1}{2}})^{n-2m}(-1)^m[w(w-1)]^m \binom{n}{2m}(2m)! + o(t^{\frac{n-2m}{2}}). \tag{4.8}$$

We can finally Taylor expand $\phi(t, w + \alpha t^{1/2})$ up to $\kappa = \lfloor \frac{n}{2} \rfloor$:

$$\phi(t, w + \alpha t^{1/2}) = ct^{n/2} \sum_{k=0}^{\kappa} (-1)^k \alpha^{n-2k}[w(w-1)]^k \frac{n!}{k!(n-2k)!} + o(t^{n/2}). \tag{4.9}$$

Absorbing $\sqrt{w(w-1)}$ into α, we recognize a variant of the n'th Hermite polynomial:

$$H_n(\alpha) = \sum_{k=0}^{\kappa} (-1)^k \frac{\alpha^{n-2k} n!}{k!(n-2k)!},$$

which is known (e.g. [20, §3.3]) to have n distinct real roots. □

Proof of "if" direction in Theorem 1.1. By Proposition 3.13, the birth-death chain with constant birth and linear death rates preserves t-stability, and we just showed that the pure quadratic death chain preserves stability (and hence t-stability). However, the latter chain is no longer a Feller process, so we cannot immediately apply Trotter's product formula – as we did with reaction-diffusion processes and independent Markov chains – to combine the two processes. Indeed, it is well known that pure quadratic death chain comes down from infinity in finite time, in the sense that $\liminf_{k\to\infty} p_t(k, 1) > 0$ for each $t > 0$ [11].

We rectify this situation by considering the Banach space $l^1(\mathbb{N})$ of absolutely summable sequences. Let $X_t^{(1)}$, $X_t^{(2)}$, and $X_t^{(3)}$ be the birth-death chains with respective rates $\{\beta_k^{(1)} = b_0,\ \delta_k^{(1)} = d_1 k\}$, $\{\beta_k^{(2)} = 0,\ \delta_k^{(2)} = d_2 k(k-1)\}$, and $\{\beta_k^{(3)} = b_0,\ \delta_k^{(3)} = d_1 k + d_2 k(k-1)\}$. With

$$P^{(i)}(t)f(x) = \sum_y f(y) P(X_t^{(i)} = x | X_0^{(i)} = y)$$

as the (adjoint) strongly continuous contraction semigroups on $l^1(\mathbb{N})$, we consider the infinitesimal generators as the l^1 limit

$$\Omega^{(i)} f = \lim_{t\downarrow 0} \frac{P^{(i)}(t)f - f}{t}.$$

See [17] for the theory of adjoint semigroups of Markov chains.

Let

$$D_0 = \{f \in l^1(\mathbb{N});\ f(x) = 0 \text{ for all but finitely many } x\},$$
$$D_e = \{f \in l^1(\mathbb{N});\ |f(x)| \le Ce^{-x}\},\ C \text{ depending only on } f, \text{ and}$$
$$\mathcal{D}(\Omega^{(i)}) = \{f \in l^1(\mathbb{N});\ \lim_{t\downarrow 0} t^{-1}(P^{(i)}(t)f - f) \text{ exists as an } l^1 \text{ limit.}\}$$

By explicit calculation, it can be seen that $D_0 \subset D_e \subset \mathcal{D}(\Omega^{(i)})$ for each i, $P^{(i)}(t) : D_0 \to D_e$, and for $f \in D_e$,

$$\Omega^{(i)} f(x) = \delta^{(i)}_{x+1} f(x+1) + \beta^{(i)}_{x-1} f(x-1) - [\beta^{(i)}_x + \delta^{(i)}_x] f(x).$$

By [9, Prop. 3.3 of Ch. 1], D_e is a core for all three generators. Also,

$$\Omega^{(1)} + \Omega^{(2)} = \Omega^{(3)} \quad \text{on } D_e,$$

so we can apply Trotter's product formula to conclude preservation of t-stability for $X_t^{(3)}$. □

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