Distributional Limits for the Symmetric Exclusion Process

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January 22, 2008

Abstract

Strong negative dependence properties have recently been proved for the symmetric exclusion process. In this paper, we apply these results to prove convergence to the Poisson and Gaussian distributions for various functionals of the process.

1 Introduction

The symmetric exclusion process on the countable set S is the Markov process η_t on $\{0,1\}^S$ with formal generator

$$\mathcal{L}f(\eta) = \sum_{\eta(x)=1,\eta(y)=0} p(x,y)[f(\eta_{x,y}) - f(\eta)],$$

where $\eta_{x,y}$ is the configuration obtained from η by interchanging the coordinates $\eta(x)$ and $\eta(y)$. Here $p(x,y) = p(y,x)$ are the transition probabilities for a symmetric, irreducible, Markov chain on S. For background on this process, see Chapter VIII of [10].

Many limit theorems of various types have been proved for this process. Examples are the central limit theorems for a tagged particle and for the flux in one dimensional systems in [1], [7], [8], [9], and [12]. In this paper, we focus on limit theorems that can now be proved using the recently obtained strong negative dependence properties of the symmetric exclusion process.

A probability measure μ on $\{0,1\}^S$ is said to be negatively associated if

$$\int fg d\mu \leq \int f d\mu \int g d\mu$$

Mathematics Subject Classification. 60K35.

Key words and phrases: Exclusion processes, negative association, negative dependence, central limit theorems, stability of polynomials.

Research supported in part by NSF Grant DMS-0301795.

for all increasing continuous functions f, g on $\{0,1\}^S$ that depend on disjoint sets of coordinates. Theorem 5.2 of [4] asserts that if the initial distribution of the symmetric exclusion process η_t is a product measure, then the distribution of η_t is negatively associated for all $t > 0$. In fact, by Proposition 5.1 of that paper, it has a stronger and even more useful property, known as strong Rayleigh.

Limit theorems for negatively associated random variables have been proved by a number of authors – see [2], [11], and [14], for example. In the case of convergence to the normal law, none of these results quite fit our setting. In our situation, there is generally no translation invariance in the covariance structure, and the sum of off-diagonal covariances is often not "little o" of the sum of variances. However, we will see in Section 2 that the strong Rayleigh property makes it quite easy to prove convergence to the Poisson and Gaussian laws, given estimates of variances and covariances. Therefore, we will not need to use these earlier results.

The first situation we will consider involves the extremal stationary distributions for the process. We recall their description – see Chapter VIII of [10]. Let

$$\mathcal{H} = \left\{ \alpha : S \to [0,1], \sum_y p(x,y)\alpha(y) = \alpha(x) \ \forall x \right\},$$

and for $\alpha \in \mathcal{H}$, let ν_α be the product measure with marginals $\nu\{\eta : \eta(x) = 1\} = \alpha(x)$. Then the limiting distribution as $t \to \infty$ of the process at time t exists if the initial distribution is ν_α; call it μ_α. The result is that the extremal stationary distributions are exactly $\{\mu_\alpha, \alpha \in \mathcal{H}\}$. If α is constant, then $\mu_\alpha = \nu_\alpha$ so we are really interested in nonconstant α's, in which case very little is known about the corresponding stationary distributions other than the marginals – $\mu_\alpha\{\eta : \eta(x) = 1\} = \alpha(x)$. If $p(x,y)$ are the transition probabilities for simple random walk on a homogeneous tree, for example, there are many such nonconstant α's, and therefore there are many extremal stationary distributions that are not of product form. We now know from the results in [4] that μ_α is negatively associated – and in fact strong Rayleigh – for each $\alpha \in \mathcal{H}$.

We will use the following notation. For $n \geq 1$, $p^{(n)}(x,y)$ are the n-step transition probabilities for the Markov chain with transition probabilities $p(x,y)$, and for $t > 0$,

$$p_t(x,y) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} p^{(n)}(x,y)$$

are the transition probabilities for the corresponding continuous time chain X_t. The Green function is given by

$$G(x,y) = \sum_{n=0}^{\infty} p^{(n)}(x,y) = \int_0^\infty p_t(x,y)\,dt.$$

The Dirichlet sum of an $\alpha \in \mathcal{H}$ is defined by

$$\Phi(\alpha) = \sum_{x,y} p(x,y)[\alpha(y) - \alpha(x)]^2.$$

This quantity is finite for many, but not all, elements of $\mathcal{H}$ if S is a regular tree, for example. A construction of a class of infinite graphs with only one end that support nonconstant $\alpha \in \mathcal{H}$ with $\Phi(\alpha) < \infty$ is constructed in [6]. In this context, we have the following results. We use $\Rightarrow$ to denote convergence in distribution.

Theorem 1. *Suppose $\alpha \in \mathcal{H}$ and $\Phi(\alpha) < \infty$. If $S_n \subset S$ satisfy*

(a) $\lim_{n\to\infty} \sup_{x\in S_n} \alpha(x) = 0$, $\lim_{n\to\infty} \sum_{x\in S_n} \alpha(x) = \lambda < \infty$,
and

(b) $\sup_{x,n} \sum_{y\in S_n} G(x,y) < \infty$,
then under μ_α,

$$\sum_{x\in S_n} \eta(x) \Rightarrow Poisson(\lambda).$$

Theorem 2. *Suppose $\alpha \in \mathcal{H}$ and $\Phi(\alpha) < \infty$. If $S_n \subset S$ satisfy*

$$\lim_{n\to\infty} \sum_{x\in S_n} \alpha(x)[1-\alpha(x)] = \infty$$

and

$$\liminf_{n\to\infty} \frac{\sum_{x\in S_n} \alpha(x)[1-\alpha(x)]}{\sup_x \sum_{y\in S_n} G(x,y)} > \Phi(\alpha). \tag{1}$$

Then under μ_α,

$$\frac{\sum_{x\in S_n} \eta(x) - \sum_{x\in S_n} \alpha(x)}{[Var_{\mu_\alpha}(\sum_{x\in S_n} \eta(x))]^{1/2}} \Rightarrow N(0,1).$$

Furthermore,

$$Var_{\mu_\alpha}\left(\sum_{x\in S_n} \eta(x)\right) \Big/ \sum_{x\in S_n} \alpha(x)[1-\alpha(x)] \tag{2}$$

(which is at most one) is bounded below by a positive constant. If the left side of (1) *is infinite, then the limit of (2) as $n \to \infty$ is 1.*

Theorems 1 and 2 will be proved in Section 3, after deriving limit theorems for general strong Rayleigh Bernoulli random variables in Section 2.

As an example of the application of Theorems 1 and 2, let S be the binary tree, and let the chain have nearest neighbor jumps with probability 1/3 each. Write $S = L \cup R$ where L, R are defined as follows: A basis edge is fixed, and its endpoints are called left and right respectively. Then L is the set of vertices that are closer to the left vertex than to the right, and $R = S\backslash L$. Each vertex x in S is assigned a level $l(x) \geq 0$, which is the distance from x to the closer of the two endpoints of the basis edge. Then $G(x,y) = 2^{-d(x,y)+1}$, where $d(x,y)$ is the distance between x and y. Therefore, $\sup_x \sum_{y:l(y)<n} G(x,y) = 3n$, $\sup_x \sum_{y\in L:l(y)<n} G(x,y) = 2n$ and $\sup_x \sum_{y\in L:l(y)=n} G(x,y) = 3 - 2^{-n}$.

For $0 \leq \lambda, \rho \leq 1$, let $\alpha \in \mathcal{H}$ be defined by

$$\alpha(x) = \begin{cases} \lambda + \frac{\rho-\lambda}{3\cdot 2^{l(x)}} & \text{if } x \in L, \\ \rho + \frac{\lambda-\rho}{3\cdot 2^{l(x)}} & \text{if } x \in R, \end{cases}$$

and put $\mu = \mu_\alpha$. Then $\Phi(\alpha) = 2(\rho-\lambda)^2/9$,

$$E^\mu \sum_{x:l(x)<n} \eta(x) = (\lambda+\rho)(2^n - 1),$$

and

$$\sum_{x:l(x)<n} Var_\mu \eta(x) = (2^n-1)[\rho(1-\rho)+\lambda(1-\lambda)] + (\lambda-\rho)^2\left[\frac{2}{3}n - \frac{4}{9}(1-2^{-n})\right].$$

It follows that for $S_n = \{x : l(x) < n\}$, the left side of (1) is infinite if

$$\rho(1-\rho)+\lambda(1-\lambda) > 0.$$

Therefore, for all choices of λ, ρ, Theorem 2 implies that

$$\frac{\sum_{x:l(x)<n}\eta(x) - (\lambda+\rho)2^n}{\sqrt{2^n}} \Rightarrow N(0, \rho(1-\rho)+\lambda(1-\lambda)). \tag{3}$$

Next take $\lambda = 0, \rho = 1$, in which case (3) has little content. Then Theorem 1 gives

$$\sum_{x\in L:l(x)=n}\eta(x) \Rightarrow Poisson(1/3).$$

If $S_n = \{x \in L : l(x) < n\}$, the left and right sides of (1) are $\frac{1}{6}$ and $\frac{2}{9}$ respectively, so (1) does not hold in this case. Nevertheless, we will see at the end of Section 3 that more careful estimates imply that

$$\frac{\sum_{x\in L:l(x)<n}\eta(x) - \frac{n}{3}}{\sigma_n} \Rightarrow N(0,1), \tag{4}$$

where σ_n^2/n is asymptotically between $\frac{23}{189}$ and $\frac{1}{3}$.

The next situation we consider was proposed in [13] as an application of the then hoped for negative dependence properties of the symmetric exclusion process. Now $S = Z^1$, and $p(x,y) = p(y-x)$, with $\sum_x |x|p(x) < \infty$. For the initial configuration, we take

$$\eta(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Since $\sum_{x<0<y} p(x,y) < \infty$, $W_t = \sum_{x>0}\eta_t(x) < \infty$ a.s. for all t.

Theorem 3. *Suppose $\sigma^2 = \sum_n n^2 p(n) < \infty$. Then*

$$\frac{W_t - EW_t}{[Var(W_t)]^{1/2}} \Rightarrow N(0,1) \tag{5}$$

as $t \to \infty$. Furthermore,

$$\lim_{t\to\infty}\frac{EW_t}{\sqrt{t}} = \frac{\sigma}{\sqrt{2\pi}},$$

and

$$\frac{Var(W_t)}{t^{1/2}}$$

is bounded above and below by positive constants.

Theorem 3 will be proved in Section 4. It seems likely that if the distribution $p(\cdot)$ is in the domain of normal attraction of a (symmetric) stable law of index $\alpha \in (1,2)$, then (5) holds with $Var(W_t)$ of order $t^{1/\alpha}$, but this has not been checked.

The major result from [4] that we use in this paper is Proposition 5.1, which asserts that the strong Rayleigh property is preserved by the evolution of a symmetric exclusion process. The proof given there uses results from earlier papers, including Obreschkoff's Theorem, which are not part of the toolkit of a typical probabilist. In Section 5, we present an elementary proof of that result in order to make the present paper essentially self-contained.

2 Limit theorems for strong Rayleigh measures

Consider a probability measure μ on $\{0,1\}^n$. Its generating polynomial is defined for $z \in \mathcal{C}^n$ by

$$Q(z) = Q(z_1, ..., z_n) = E^\mu \prod_{i=1}^n z_i^{\eta(i)}.$$

The measure μ is said to be Rayleigh if

$$\frac{\partial Q}{\partial z_i}(z)\frac{\partial Q}{\partial z_j}(z) \geq Q(z)\frac{\partial^2 Q}{\partial z_i \partial z_j}(z), \quad i \neq j, \tag{6}$$

for all $z \in R_+^n$, and to be strong Rayleigh if (6) holds for all $z \in R^n$. Note that when $z = (1,1,...,1)$, (6) says that $\eta(i)$ and $\eta(j)$ are negatively correlated under μ. By Theorem 4.9 of [4], the strong Rayleigh property implies negative association.

Brändén ([5]) proved that the strong Rayleigh property is equivalent to the following property, which is known as stability: $Q(z) \neq 0$ if z_i has strictly positive imaginary part for each i. This is the key to the following representation.

Proposition 4. *Suppose μ is strong Rayleigh. Then there exist independent Bernoulli random variables ζ_i with parameters p_i so that the distribution of $\sum_i \eta(i)$ under μ is the same as that of $\sum_i \zeta_i$.*

Proof. Setting $z_i \equiv w$ in the expression for Q, we see that the polynomial in one variable

$$Q^*(w) = Q(w, w, ..., w) = E^\mu w^{\sum_i \eta(i)}$$

has no roots with positive imaginary part, and therefore all of its roots are real. Since $Q^*(1) = 1$, it follows that Q^* can be written in the form

$$Q^*(w) = \prod_i (w + w_i) \Big/ \prod_i (1 + w_i)$$

where the w_i are real. Since $Q^*(-w_i) = 0$, we see that $w_i \geq 0$ for each i. Letting $p_i = 1/(1 + w_i)$, this becomes

$$Q^*(w) = \prod_i \big[p_i w + (1 - p_i)\big],$$

which is the generating polynomial for $\sum_i \zeta_i$. □

Using this result, it is easy to extend the classical limit theorems to triangular arrays of strong Rayleigh random variables.

Proposition 5. *Suppose the Bernoulli random variables $\{\eta_n(x)\}$ are strong Rayleigh for each n.*

(a) If $\lim_{n\to\infty} \sum_x E\eta_n(x) = \lambda$, $\lim_{n\to\infty} \sum_x [E\eta_n(x)]^2 = 0$, and

$$\lim_{n\to\infty} \sum_{x \neq y} Cov(\eta_n(x), \eta_n(y)) = 0,$$

then

$$\sum_x \eta_n(x) \Rightarrow Poisson(\lambda).$$

(b) If $\lim_{n\to\infty} Var(\sum_x \eta_n(x)) = \infty$, *then*

$$\frac{\sum_x \eta_n(x) - E\sum_x \eta_n(x)}{\sqrt{Var(\sum_x \eta_n(x))}} \Rightarrow N(0,1).$$

Proof. Using Proposition 4, let $\zeta_{n,i}$ be Bernoulli random variables that are independent in i for each n, and have the property that

$$\sum_x \eta_n(x) \quad \text{and} \quad \sum_i \zeta_{n,i}$$

have the same distribution for each n. It suffices to show that the conditions for Poisson and normal convergence hold for the array $\zeta_{n,i}$. But this is immediate from the assumptions and the following identities:

$$\sum_x E\eta_n(x) = \sum_i E\zeta_{n,i}$$

and

$$\begin{aligned} Var\left(\sum_x \eta_n(x)\right) =& \sum_x Var(\eta_n(x)) + \sum_{x\neq y} Cov(\eta_n(x), \eta_n(y)) \\ =& Var(\sum_i \zeta_{n,i}) = \sum_i Var(\zeta_{n,i}). \end{aligned}$$

□

Remark 6. *Proposition 5(a) is a consequence of Theorem 11 in [11] or Theorem 3D in [2] The latter also gives bounds on the total variation distance from the Poisson distribution. We note that under the strong Rayleigh assumption we are making, the proof is very simple.*

3 The stationary distributions

We begin the proofs of Theorems 1 and 2 by obtaining a bound on the covariances for the measure $\mu = \mu_\alpha$. Let U and $U(t)$ be the generator and semigroup for the motion of two independent copies of the Markov chain with transition probabilities $p_t(x,y)$, and V and $V(t)$ be the generator and semigroup for the motion of two copies of that Markov chain with the exclusion interaction. Then by (1.28) and (1.29) on page 373 of [10],

$$E^\mu \eta(x)\eta(y) = \lim_{t\to\infty} V(t)f(x,y), \quad x \neq y,$$

where $f(x,y) = \alpha(x)\alpha(y)$. Since $U(t)f(x,y) = f(x,y)$, the integration by parts formula gives

$$\begin{aligned} -Cov_\mu(\eta(x),\eta(y)) =& f(x,y) - \lim_{t\to\infty} V(t)f(x,y) \\ =& \lim_{t\to\infty}\Big[U(t)f(x,y) - V(t)f(x,y)\Big] \\ =& \lim_{t\to\infty}\int_0^t V(s)[U-V]U(t-s)f(x,y)ds \\ =& \int_0^\infty V(s)[U-V]f(x,y)ds \end{aligned} \tag{7}$$

for $x \neq y$. From page 366 of [10], we see that for $x \neq y$,

$$(U - V)f(x, y) = p(x, y)[f(x, x) + f(y, y) - 2f(x, y)] = p(x, y)[\alpha(x) - \alpha(y)]^2. \tag{8}$$

Define $\Delta(x, y) = p(x, y)[\alpha(x) - \alpha(y)]^2$ for all x, y.

Let $g_n(x, y) = 1_{S_n \times S_n}(x, y)$. This is a positive definite function. Therefore, using the symmetry of $V(s)$, the fact that $\Delta(x, y) = 0$ if $x = y$, and Proposition 1.7 on page 366 of [10], we see that

$$\begin{aligned} - \sum_{x,y \in S_n; x \neq y} Cov_\mu(\eta(x), \eta(y)) &= \int_0^\infty \sum_{x \neq y} g_n(x, y) V(s) \Delta(x, y) ds \\ &= \int_0^\infty \sum_{x,y} \Delta(x, y) V(s) g_n(x, y) ds \\ &\leq \int_0^\infty \sum_{x,y} \Delta(x, y) U(s) g_n(x, y) ds \\ &= \sum_{x,y} \Delta(x, y) \int_0^\infty P^x(X_s \in S_n) P^y(X_s \in S_n) ds, \end{aligned} \tag{9}$$

where X_s is the Markov chain with transition probabilities $p_s(x, y)$.

Proof of Theorem 1. Given the strong Rayleigh property of μ and Proposition 5(a), it suffices to check that

$$\lim_{n \to \infty} \sum_{x,y \in S_n; x \neq y} Cov_\mu(\eta(x), \eta(y)) = 0,$$

and therefore we need to check that the right side of (9) tends to zero. To do so, note that

$$\int_0^\infty P^x(X_s \in S_n) ds = \sum_{u \in S_n} G(x, u) \leq C_1 \tag{10}$$

for some constant C_1 by assumption (b). Next, since

$$\sup_{s>0} e^{-s} \frac{s^k}{k!} = e^{-k} \frac{k^k}{k!} \leq \frac{C_2}{\sqrt{k}}$$

for some constant C_2, we have for any $N \geq 1$,

$$P^y(X_s \in S_n) \leq \sum_{k=0}^{N} \sum_{v \in S_n} p^{(k)}(y, v) + \frac{C_1 C_2}{\sqrt{N}}, \tag{11}$$

so that for each y

$$\lim_{n \to \infty} \sup_{s>0} P^y(X_s \in S_n) = 0. \tag{12}$$

To see this, note that by the first part of assumption (a) of the theorem, the sets S_n eventually do not intersect any finite subset of S. Therefore, for fixed N and y, the first term on the right of (11) tends to zero as $n \to \infty$. We then conclude that the right side of (9) tends to zero by (10), (12), the finiteness of $\Phi(\alpha)$ and the dominated convergence theorem. $\square$

Proof of Theorem 2. By Proposition 5(b), we need to show that

$$\lim_{n\to\infty} Var_\mu\left(\sum_{x\in S_n}\eta(x)\right)=\infty. \tag{13}$$

By (9),

$$-\sum_{x,y\in S_n;x\neq y} Cov_\mu(\eta(x),\eta(y))\le \Phi(\alpha)\sup_x\sum_{u\in S_n}G(x,u).$$

Therefore, (13) follows from the assumptions in the theorem. □

We now return to the example discussed in the introduction – the simple random walk on the binary tree with

$$\alpha(x)=\begin{cases}\frac{1}{3\cdot 2^{l(x)}} & \text{if } x\in L,\\ 1-\frac{1}{3\cdot 2^{l(x)}} & \text{if } x\in R,\end{cases}$$

and $S_n=\{x\in L: l(x)<n\}$. We will see that Proposition 5(b) applies, even though (1) fails. In order to do so, we will use the structure of the problem to estimate the right side of (9) more carefully.

First note that $l(X_t)$ is a Markov chain on the nonnegative integers with drift $\frac{1}{3}$. Therefore

$$\frac{l(X_t)}{t}\to\frac{1}{3}\quad a.s. \tag{14}$$

by the strong law of large numbers. Secondly, if $\beta,\gamma>0$ are chosen so that

$$e^{-\gamma}+e^{+\gamma}=3(1-\beta),$$

then

$$e^{\beta t-\gamma l(X_t)}$$

is a supermartingale, so that

$$1\ge E^x e^{\beta nt-\gamma l(X_{nt})}\ge e^{\beta nt-\gamma n}P^x(l(X_{nt})<n).$$

This implies

$$P^x(l(X_{nt})<n)\le e^{n(\gamma-\beta t)}\le e^{\gamma-\beta t}$$

if $n\ge 1$ and $t\ge\gamma/\beta$, which provides the domination in the following computation.

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{x,y}\Delta(x,y)\int_0^\infty P^x(X_s\in S_n)P^y(X_s\in S_n)ds\\ =\lim_{n\to\infty}\sum_{x,y}\Delta(x,y)\int_0^\infty P^x(X_{nt}\in S_n)P^y(X_{nt}\in S_n)dt\\ =3\sum_{x,y}\Delta(x,y)[1-\alpha(x)][1-\alpha(y)].\end{aligned} \tag{15}$$

For the final step above, note that the integrand in the middle line tends to zero if $t>3$ by (14), while if $t<3$,

$$\lim_{n\to\infty}P^x(X_{nt}\in S_n)=P^x(X_s\in L\text{ eventually})=1-\alpha(x).$$

Next, we compute the right side of (15).

$$
\begin{aligned}
&3\sum_{x,y}\Delta(x,y)[1-\alpha(x)][1-\alpha(y)] = \sum_{d(x,y)=1}[\alpha(x)-\alpha(y)]^2[1-\alpha(x)][1-\alpha(y)] \\
=&2\left[\frac{2}{3^4}+\sum_{n=1}^{\infty}2^n\frac{1}{(3\cdot 2^n)^2}\frac{1}{3\cdot 2^n}\frac{1}{3\cdot 2^{n-1}}+\sum_{n=1}^{\infty}2^n\frac{1}{(3\cdot 2^n)^2}\left(1-\frac{1}{3\cdot 2^n}\right)\left(1-\frac{1}{3\cdot 2^{n-1}}\right)\right] \\
=&\frac{40}{189}<\frac{1}{3}.
\end{aligned}
$$

Combining this with (9) and (15), we see that

$$
\liminf_{n\to\infty}\frac{Var\Big(\sum_{x\in S_n}\eta(x)\Big)}{n}>0,
$$

so that Proposition 5(b) gives (4).

4 Limit theorems in one dimension

In this section, we will prove Theorem 3. We need to consider the first and second moments of W_t. By duality,

$$
\begin{aligned}
EW_t =&E^\eta\sum_{x>0}\eta_t(x)=\sum_{x>0}P^\eta(\eta_t(x)=1) \\
=&\sum_{x>0}P^x(X_t\le 0)=\sum_{y\le 0<x}p_t(x,y) \\
=&\sum_{n=1}^{\infty}np_t(0,n)=E^0X_t^+.
\end{aligned}
\tag{16}
$$

Similarly,

$$
\sum_{x>0}\left[P^\eta(\eta_t(x)=1)\right]^2=\sum_{x>0;u,v\le 0}p_t(x,u)p_t(x,v)=E^{(0,0)}\min(X_t^+,Y_t^+),
\tag{17}
$$

where Y_t is an independent copy of X_t.

For the covariances, we proceed as in Section 3, but this time with $f(x,y)=1_{\{x,y\le 0\}}$ and $g(x,y)=1_{\{x,y>0\}}$. For $x\ne y$, $(U-V)f(x,y)=p(x,y)[f(x,x)+f(y,y)-2f(x,y)]$, so for $x\ne y$,

$$
\begin{aligned}
(U-V)U(s)f(x,y) =&p(x,y)[U(s)f(x,x)+U(s)f(y,y)-2U(s)f(x,y)] \\
=&p(x,y)[P^x(X_s\le 0)-P^y(X_s\le 0)]^2 \\
=&p(x,y)[P^0(X_s\ge x)-P^0(X_s\ge y)]^2=\Delta_s(x,y),
\end{aligned}
$$

where we define for all x,y,

$$
\Delta_s(x,y)=\begin{cases} p(x,y)[P^0(x\le X_s<y)]^2 & \text{if } x<y, \\ p(x,y)[P^0(y\le X_s<x)]^2 & \text{if } y<x, \\ 0 & \text{if } y=x. \end{cases}
$$

Then letting

$$K(t) = -\sum_{x,y>0;x\neq y} Cov(\eta_t(x), \eta_t(y)),$$

we see that

$$\begin{aligned}
K(t) &= \sum_{x,y>0;x\neq y} [U(t) - V(t)]f(x,y) = \sum_{x,y>0;x\neq y} \int_0^t V(t-s)(U-V)U(s)f(x,y)ds \\
&= \int_0^t \sum_{x\neq y} g(x,y)V(t-s)\Delta_s(x,y)ds = \int_0^t \sum_{x,y} \Delta_s(x,y)V(t-s)g(x,y)ds \\
&\leq \int_0^t \sum_{x,y} \Delta_s(x,y)U(t-s)g(x,y)ds \\
&= \int_0^t \sum_{x,y} \Delta_s(x,y)P^0(X_{t-s} < x)P^0(X_{t-s} < y)ds.
\end{aligned} \tag{18}$$

Let $\rho(t,x) = P^0(X_t < x)$. Since $\Delta_s(x,y) = \Delta_s(y,x) = \Delta_s(1-x, 1-y)$, we may continue (18) by writing

$$\begin{aligned}
K(t) \leq& \frac{1}{2} \int_0^t \sum_{x,y} \Delta_s(x,y)[\rho(t-s,x)\rho(t-s,y) + \rho(t-s,1-x)\rho(t-s,1-y)]ds \\
=& \int_0^t \sum_{x<y} p(x,y)[P^0(x \leq X_s < y)]^2\gamma(t-s,x,y)ds,
\end{aligned} \tag{19}$$

where

$$\gamma(t,x,y) = \rho(t,x)\rho(t,y) + \rho(t,1-x)\rho(t,1-y).$$

Note that since $\rho(t,x) + \rho(t,1-x) = 1$, it follows that $\gamma(t,x,y) \leq 1$. Now let

$$\Gamma(t,n,u,v) = \sum_{x:x\leq u,v<x+n} \gamma(t,x,x+n) \leq (n - |v-u|)^+. \tag{20}$$

Then using the symmetry and translation invariance of $p_s(x,y)$,

$$\begin{aligned}
K(t) &\leq \int_0^t \sum_{n=1}^\infty p(n) \sum_{u,v} p_s(0,u)p_s(0,v)\Gamma(t-s,n,u,v)ds \\
&= \int_0^t \sum_{n=1}^\infty p(n)E^0\Gamma(t-s,n,X_s,X_s - X_{2s})ds \\
&\leq \int_0^t \sum_{n=1}^\infty p(n)E^0(n - |X_{2s}|)^+ds.
\end{aligned} \tag{21}$$

Now assume that $\sigma^2 = \sum_n n^2p(n) < \infty$, and write

$$E^0(n - |X_{2s}|)^+ = \sum_{k=-n}^{n} (n-|k|)p_{2s}(0,k) \leq n^2p_{2s}(0,0),$$

where the inequality comes from

$$[p_{2s}(0,k)]^2 = \left[\sum_j p_s(0,j)p_s(j,k)\right]^2 \le \sum_j [p_s(0,j)]^2 \sum_j [p_s(j,k)]^2 = [p_{2s}(0,0)]^2.$$

Therefore,

$$\limsup_{t\to\infty} \frac{1}{\sqrt{t}} \int_0^t \sum_{n>N} p(n) E^0 (n - |X_{2s}|)^+ ds \le \frac{1}{\sigma\sqrt{\pi}} \sum_{n>N} n^2 p(n). \tag{22}$$

by the local central limit theorem. For the terms corresponding to small n, we have given up too much in using the inequality in (20). To handle these terms, first note that for fixed k, conditionally on $|X_{2s}| = k$, $X_s/\sqrt{s}$ is asymptotically normal with mean 0 and variance $\sigma^2/2$. Now take fixed k, n with $0 \le k \le n$. Then

$$\begin{aligned} E^0[\Gamma(t-s,n,X_s,X_s-X_{2s}) \mid X_{2s}=k] =& E^0\left[\sum_{X_s-n<x\le X_s-k} \gamma(t-s,x,x+n)\middle| X_{2s}=k\right] \\ =& E^0\left[\sum_{y=k}^{n-1} \gamma(t-s,X_s-y,X_s-y+n)\middle| X_{2s}=k\right]. \end{aligned}$$

Write

$$\begin{aligned} E^0[\gamma(t-s,X_s-y,&X_s-y+n) \mid X_{2s}=k] = P^0(Y_{t-s} < X_s - y, Z_{t-s} < X_s - y + n \mid X_{2s} = k) \\ &+ P^0(Y_{t-s} < 1 - X_s + y, Z_{t-s} < 1 - X_s + y - n \mid X_{2s} = k), \end{aligned}$$

where X_t, Y_t, Z_t are independendent copies of the random walk. If $s, t \to \infty$ with $s/r \to r \in (0,1)$, then the above expression converges to

$$\begin{aligned} &P\left(N_2 \le \sqrt{\frac{r}{2(1-r)}} N_1, N_3 \le \sqrt{\frac{r}{2(1-r)}} N_1\right) \\ &+P\left(N_2 \le -\sqrt{\frac{r}{2(1-r)}} N_1, N_3 \le -\sqrt{\frac{r}{2(1-r)}} N_1\right), \end{aligned}$$

where N_1, N_2, N_3 are independent normally distributed random variables with mean zero and variance 1. Call this expression $h(r)$. Note that $h(r) < 1$ for $r < 1$, so that

$$H = \int_0^1 \frac{h(r)}{2\sqrt{r}} dr < 1.$$

Passing to the limit in (21) we see that

$$\limsup_{t\to\infty} \frac{K(t)}{\sqrt{t}} \le \frac{H}{\sigma\sqrt{\pi}} \sum_{n=1}^{N} n^2 p(n) + \frac{1}{\sigma\sqrt{\pi}} \sum_{n>N} n^2 p(n).$$

Taking $N \to \infty$ gives

$$\limsup_{t\to\infty} \frac{K(t)}{\sqrt{t}} \le \frac{H\sigma}{2\sqrt{\pi}}.$$

Recalling from (16) and (17) that

$$Var(W_t) = E^0 X_t^+ - E^0 \min(X_t^+, Y_t^+) - K(t),$$

we see that

$$\limsup_{t\to\infty} t^{-1/2} Var(W_t) \leq \frac{\sigma}{2\sqrt{\pi}}$$

and

$$\liminf_{t\to\infty} t^{-1/2} Var(W_t) \geq \frac{(1-H)\sigma}{2\sqrt{\pi}} > 0.$$

So, Theorem 3 follows from Proposition 5(b) and the strong Rayleigh property of $\{\eta_t(x), x > 0\}$.

5 Stability and the symmetric exclusion process

In this section, we present an elementary proof of the basic fact needed to prove preservation of stability by the symmetric exclusion process. A similar proof was obtained independently by [3]

A polynomial $Q(z_1, ..., z_n)$ with complex coefficients is said to be stable if $Q(z_1, ..., z_n) \neq 0$ whenever $\Im(z_i) > 0$ for each i. The key result needed to show that the generating polynomial of the distribution a symmetric exclusion process at time t is stable whenever this is the case at time 0 is the following.

Theorem 7. *Suppose the multi-affine polynomial Q is stable. Then so is the polynomial Q_p for $0 \leq p \leq 1$, where*

$$Q_p(z_1, ..., z_n) = pQ(z_1, ..., z_n) + (1-p)Q(z_2, z_1, z_3, ..., z_n).$$

The proof of Theorem 7 is based on the following characterization of stability for multi-affine polynomials in two variables.

Proposition 8. *Suppose $h(z, w) = a + bz + cw + dzw$, where a, b, c, d are complex, and not all zero. Then h is stable if and only if*

$$\Re(b\bar{c} - a\bar{d}) \geq |bc - ad|, \ \Im(a\bar{b}) \geq 0, \ \Im(a\bar{c}) \geq 0, \ \Im(b\bar{d}) \geq 0, \ \Im(c\bar{d}) \geq 0. \tag{23}$$

Proof. If $b = c = d = 0$, the h is automatically stable, since then $a \neq 0$. If $d = 0, b \neq 0$, then $h(z, w) = 0$ iff

$$z = -\frac{(a + cw)\bar{b}}{|b|^2},$$

so stability is equivalent to $\Im(w) > 0 \Rightarrow \Im(z) \leq 0$, where

$$\Im(z) = -\frac{\Im(a\bar{b}) + \Re(c\bar{b})\Im(w) + \Im(c\bar{b})\Re(w)}{|b|^2}.$$

Therefore, since $\Re(w)$ is arbitrary, stability is equivalent to $c\bar{b} \geq 0$ and $\Im(a\bar{b}) \geq 0$, and these imply $\Im(a\bar{c}) = b\bar{c}\Im(a\bar{b})/|b|^2 \geq 0$ in this case.

So, we may now assume that $d \neq 0$. Solving for z, we see that $h(z, w) = 0$ iff

$$b + dw = 0, \quad a + cw = 0 \tag{24}$$

or

$$b + dw \neq 0, \quad z = -\frac{a + cw}{b + dw} = -\frac{a\bar{b} + c\bar{b}w + a\bar{d}\bar{w} + c\bar{d}|w|^2}{|b + dw|^2}. \tag{25}$$

Case (24) occurs iff

$$w = -\frac{b}{d} = -\frac{b\overline{d}}{|d|^2}, \quad bc = ad. \tag{26}$$

Therefore, if $bc \neq ad$, stability is equivalent to the statement

$$\Im(w) \geq 0 \Rightarrow \Im(a\overline{b}) + \Re(w)\Im(c\overline{b} + a\overline{d}) + \Im(w)\Re(c\overline{b} - a\overline{d}) + |w|^2\Im(c\overline{d}) \geq 0, \tag{27}$$

while if $bc = ad$, stability is equivalent to this, together with $\Im(b\overline{d}) \geq 0$.

Letting w be real, we see that for (27) to hold, we need

$$\Im(a\overline{b}) \geq 0, \quad \Im(c\overline{d}) \geq 0, \quad [\Im(c\overline{b} + a\overline{d})]^2 \leq 4\Im(a\overline{b})\Im(c\overline{d}).$$

Note that

$$4\Im(a\overline{b})\Im(c\overline{d}) - [\Im(c\overline{b} + a\overline{d})]^2 = [\Re(c\overline{b} - a\overline{d})]^2 - |bc - ad|^2.$$

Minimizing the expression in (27) over $\Re(w)$, we see that we also need $\Re(c\overline{b} - a\overline{d}) \geq 0$ if $\Im(c\overline{d}) = 0$, and

$$\Im(a\overline{b}) - \frac{[\Im(c\overline{b} + a\overline{d})]^2}{4\Im(c\overline{d})} + \Re(c\overline{b} - a\overline{d})t + \Im(c\overline{d})t^2 \geq 0, \quad t \geq 0 \tag{28}$$

if $\Im(c\overline{d}) > 0$. Since the discriminant of this quadratic is

$$[\Re(c\overline{b} - a\overline{d})]^2 + [\Im(c\overline{b} + a\overline{d})]^2 - 4\Im(a\overline{b})\Im(c\overline{d}) = |bc - ad|^2 \geq 0,$$

if $\Im(c\overline{d}) > 0$, (28) is always true if $bc = ad$, while if $bc \neq ad$, (28) is equivalent to $\Re(c\overline{b} - a\overline{d}) \geq 0$. Putting these observations together, noting that stability is not changed if the roles of b and c reversed, completes the proof. $\square$

Proof of Theorem 7. We need to show that $Q_p(z_1, ..., z_n) \neq 0$ whenever $\Im(z_i) > 0$ for $i = 1, ..., n$. To to so, fix $z_3, ...z_n$ with $\Im(z_i) > 0$ for $i = 3, ..., n$, and write

$$h(z, w) = Q(z, w, z_3, ..., z_n).$$

Then h is of the form considered in Proposition 8, and we must show that if (23) holds for a given a, b, c, d, then it also holds with b and c replaced by

$$b(p) = pb + (1 - p)c, \quad c(p) = pc + (1 - p)b.$$

This follows from

$$\Im(a\overline{b(p)}) = p\Im(a\overline{b}) + (1 - p)\Im(a\overline{c}), \quad \Im(a\overline{c(p)}) = p\Im(a\overline{c}) + (1 - p)\Im(a\overline{b}),$$
$$\Im(b(p)\overline{d}) = p\Im(b\overline{d}) + (1 - p)\Im(c\overline{d}), \quad \Im(c(p)\overline{d}) = p\Im(c\overline{d}) + (1 - p)\Im(b\overline{d}),$$

and

$$\Re(b(p)\overline{c(p)} - a\overline{d}) = \Re(b\overline{c} - a\overline{d}) + p(1 - p)|b - c|^2,$$
$$b(p)c(p) - ad = (bc - ad) + p(1 - p)(b - c)^2,$$

so that

$$\Re(b(p)\overline{c(p)} - a\overline{d}) - |b(p)c(p) - ad| \geq \Re(b\overline{c} - a\overline{d}) - |bc - ad|$$

by the triangle inequality. $\square$

Corollary 9. *Suppose the generating polynomial of the initial distribution of a symmetric exclusion process on a finite set S is stable. Then the same is true at time $t > 0$.*

Proof. View the process in terms of stirrings. In other words, for each pair $x, y \in S$ interchange $\eta(x)$ and $\eta(y)$ at Poisson times at a certain rate. If stirrings are applied at only one pair of sites, the generating polynomial of the distribution at time t is of the form Q_p given in the statement of Theorem 5, where Q is the generating polynomial of the initial distribution. For a general exclusion process, the distribution at time t can be obtained as a limit of that obtained by successively applying stirrings at different pairs of sites. □

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