Comments and Corrections for Continuous Time Markov Processes: An Introduction

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(Thanks to N. Gantert and T. Seppalainen for many of these.)

Page xi. One occurrence of "sections" has a typo.

Page 5. Definition 1.10 should read as follows: A stochastic process $(X(t), t \geq 0)$ has stationary increments if the joint distribution of $(X(t_{j+1} + s) - X(t_j + s), 1 \leq j < n)$ does not depend on s whenever $0 \leq t_1 < t_2 < \cdots < t_n$ and $n \geq 1$. It has independent increments if ....

(Of course, if the increments are independent, this version of the definition agrees with the previous one.)

Page 7. To avoid referring to a possibly nonmeasurable set, change Definition 1.14 to read $P(A) = 1$ for some measurable set A so that $X(t, \omega)$ is a continuous function of t for every $\omega \in A$.

Page 18. In Theorem 1.40(a), remove the word "hence".

Page 22. The continuity of (1.13) is obvious, since the f_m's are bounded and continuous. The proof given in the book is a holdover from an earlier version, in which the f_m's were just assumed to be bounded and measurable.

Page 23, line 13. ... of bounded random variables Y for which the mapping ...

Page 27. The second and third sentences in the proof of Theorem 1.52 are unnecessary. The display in the proof shows, as mentioned there, that $P^x(A)$ is a continuous function of x. Setting $x = 0$ in that display then gives the conclusion of the theorem.

Page 28. In the proof of Proposition 1.56, replace Q by Q^+.

Page 29. In the proof of Proposition 1.60, there is a typo: Furthermore.

Page 33. In the first display, replace E by E^x.

Page 34. In the definition of τ in the proof of Theorem 1.74, change $X(s) = a$ to $X(s) > a$.

Page 36. In the proof of Corollary 1.77, nothing is said about why the joint density exists. There are several ways to see this. Perhaps the simplest is to use the fact that the joint distribution function, being increasing in both variables, is a.e. differentiable. Since the derivative integrates to 1, it follows that the joint distribution is absolutely continuous and has the given density.

Page 37. Suggestion for Exercise 1.81 (a): Of course, $Y_t(\omega)$ should be an indicator, and for the application of the strong Markov Property, it should satisfy

$$Y_{\tau_a}(\theta_{\tau_a}\omega) = 1 \quad \text{if and only if} \quad \tau_{a+b}(\omega) < \infty$$

on the event $\tau_a < \infty$. Rewrite the statement $\tau_{a+b}(\omega) < \infty$ in such a way that it involves $\theta_{\tau_a}\omega$ and τ_a, and then replace these expressions by ω and t respectively to define $Y_t(\omega)$. Then compute $E^x Y_t$, and see what it becomes when $x = t + a$.

Page 41. Following the proof of Theorem 1.93, insert the following. Exercise. An immediate consequence of Theorem 1.93 is that if τ is a stopping time and $M(t)$ is a right continuous martingale, then $M(\tau \wedge t)$ is a martingale with respect to the filtration $\mathcal{F}_{\tau\wedge t}$. Show that this is also the case with respect to the (larger) filtration $\mathcal{F}_t$.

Page 57, line -2. Change Ω to $(\Omega, \mathcal{F})$.

Page 59, Exercise 2.5. The Poisson process and the discrete time chain should of course be independent.

Page 60, Exercise 2.8(b). Use Laplace transforms (or any other technique) to check that ...

Pages 73-74. Here is another way of making this construction that avoids conditioning: For $x \in S$ and $i \geq 1$, let $\tau_{x,i}$ be independent exponential random variables, with $\tau_{x,i}$ having parameter $c(x)$. Then use $\tau_{x,i}$ for the holding time for the ith visit by the discrete time chain to x. This may make some arguments more transparent later. See the proof of Theorem 2.50, for example.

Page 75, Exercise 2.31. Make the current exercise part (b), and add the following part (a):
(a) Let $\tau_0, \tau_1, \dots$ be independent exponentially distributed random variables with paramters $c_0, c_1, \dots$. Define $N(t)$ as on the top of page 74, and $N^*(t)$ in the same way, except that now the parameters of the τ_i's are $c_k, c_{k+1}, \dots$. Show that for $s, t > 0$,

$$P(N(t) = k, N(t+s) = k+l) = P(N(t) = k)P(N^*(s) = l).$$

Page 76, Exercise 2.35. $\alpha > 0$.

Page 78, Theorem 2.39. Add the hypothesis $\sum_{z \neq y} q(z, y) < \infty$.

Page 85, top line. This again corresponds to a non-explosive Markov chain. It is recurrent by Theorem 2.50. Letting ...

After display (2.48), add: To see this, multiply the forward equation

$$\frac{d}{dt} p_t(k, l) = \sum_j p_t(k, j) q(j, l)$$

by l and sum.

Page 89. On line -9, change to $f(k) = \theta^k$ is harmonic .... On line -11, change to $\psi(1) = 0$, and $\psi'(1) = m - 1 > 0$.

Page 92, Definition 3.1(b). Change Ω to $(\Omega, \mathcal{F})$.

Page 93, Exercise 3.2. Show that (3.2) and (3.3) imply ....

Page 93, Definition 3.4(e). Add parenthetically that in particular, $f_n \to 1$ pointwise.

Page 93. The proof of the contraction property mentioned in the paragraph following Definition 3.4 is easy when S is compact, but less so in the non-compact case. The idea in this case is the following. For fixed t and x, the mapping

$$f \to T(t)f(x)$$

is a bounded linear functional on $C(S)$. By the Riesz representation theorem, there is a finite measure $\mu_{t,x}$ on S that represents it in the sense that

$$T(t)f(x) = \int f d\mu_{t,x}.$$

Applying this to f_n and letting $n \to \infty$ shows that $\mu_{t,x}$ is a probability measure. This gives the contraction property. (Note that in part (e) of the definition, $t = 0$ is included; the functions f_n, which do not depend on t, converge pointwise to 1.)

Page 95, Exercise 3.8. ... transition function for a Markov chain on a ...

Page 95. In Exercise 3.9, "...for a death process with resurrection, given by..." Also, add a third part: (c) Under what conditions on the death rates is $T(t)$ a probability semigroup on the Banach space of functions on S with limits at infinity?

Page 99. At the end of the proof of Theorem 3.15, add: This is a consequence of the Riesz Representation Theorem and the Bounded Convergence Theorem; a uniformly bounded sequence of functions on S that converges pointwise converges weakly.

Page 102. In Exercise 3.21, the state space is intended to be R^1.

Page 110. Following the proof of Proposition 3.30, add the following:

Corollary. If $\mathcal{L}$ satisfies (a), (b), and (d) of Definition 3.12 and the weaker form of (c) that states only that $\mathcal{R}(I - \lambda\mathcal{L})$ is dense in $C(S)$ for sufficiently small positive λ, then $\overline{\mathcal{L}}$ is a probability generator.

Example. Suppose S is countable, and $\mathcal{L}f = c(Pf - f)$, with $c > 0$, P as in Exercise 2.5 and $p(x, y)$ satisfying

$$\lim_{x\to\infty} p(x, y) = 0 \quad \text{for every } y.$$

(This latter property is needed for $\mathcal{L}$ to have values in $C(S)$.) Then $\mathcal{L}$ is a probability generator. To check property (c), use Proposition 3.22, while to check (d), let f_n be the indicator function of S_n, where S_n is finite and $S_n \uparrow S$.

Page 111, lines -8 and -9. Replace E by E^x.

Page 114, following display (3.29). Add the following: In the last step, we have used the fact that

$$\int_0^n - \int_t^{n+t} = \int_0^t - \int_n^{n+t}.$$

Page 120, line -5. There is a missing overline on the generator.

Page 122, line 3. Add: and therefore is L_1−bounded

Page 122, line -9. Replace the last two X's by $\tilde{X}$ and add that $\tilde{X}$ is an independent copy of X.

Page 124, line -10. Replace the last two X's by $\tilde{X}$ and add that $\tilde{X}$ is an independent copy of X.

Page 139, in several places, R should be $\mathcal{R}$.

Page 147, line -3. The brackets are misplaced in the last term. It should read $-[\mathcal{L}T(t)f][T(t)g]$.

Page 150, line 11. Remove the parentheses around $|A|$.

Page 153, line 6. The $\eta(x)$ should be $\eta_t(x)$.

Page 167, line 8. The last 0 should be an $\emptyset$.

Page 181, line 9. The middle expression is missing brackets. It should read $|E[h(A_t)-h(B_t)]|$.

Page 181, line -3. The statement is clearer if it is changed to: ... stationary distribution μ_α for the process that satisfies....

Page 198. The square brackets on the left side of (5.10) should be absolute values.

Page 199, Exercise 5.7. Warning – $A(t)$ is not necessarily bounded!

Page 200. Before the statement of Theorem 5.10, add the following: The assumption that $Y_i \in \mathcal{F}_{t_i}$ (as opposed to $\mathcal{F}_{t_{i+1}}$, for example) is needed to ensure that the integral is an adapted process. The reason for considering left continuous integrands will be seen in the proof of Proposition 5.13.

Pages 200–202. The proof of Theorem 5.10 given there is correct. However, the version given below may be easier to follow.

Let $\Delta_i(t) = Y_i[M(t \wedge t_{i+1}) - M(t \wedge t_i)]$, and $\Delta(t) = Y[M(t \wedge b) - M(t \wedge a)]$ where $a < b$ and $Y \in \mathcal{F}_a$, which is a generic form of this. Then

$$M^*(t) = \sum_{i=1}^{\infty} \Delta_i(t),$$

so to show that $M^*(t)$ is a martingale, it suffices to check that $\Delta(t)$ is one. If $a \le s < t \le b$,

$$E[\Delta(t) - \Delta(s) \mid \mathcal{F}_s] = Y E[M(t) - M(s) \mid \mathcal{F}_s] = 0, \tag{1}$$

since M is a martingale. Since $\Delta(t)$ is independent of t for $t \ge b$, the left side of (1) is zero for all $a \le s < t$. Putting $s = a$ in (1) gives

$$E[\Delta(t) - \Delta(a) \mid \mathcal{F}_a] = 0 \quad \text{for} \quad t > a. \tag{2}$$

If $s < a$, condition (2) with respect to $\mathcal{F}_s$ to get

$$E[\Delta(t) - \Delta(a) \mid \mathcal{F}_s] = 0.$$

Since $\Delta(s) = 0$ for $s \le a$, the proof is complete.

Take $i < j$ and $s < t$. Since $\Delta_i(t) \in \mathcal{F}_{t \wedge t_{i+1}}$, $Y_j[\Delta_i(t) - \Delta_i(s)] \in \mathcal{F}_{s \vee t_j}$. Also,

$$E[M(t \wedge b) - M(t \wedge a) - M(s \wedge b) + M(s \wedge a) \mid \mathcal{F}_{s \vee a}] = 0.$$

If $s \ge a$, this is just the martingale property checked above (with $Y \equiv 1$), while if $s < a$, it becomes

$$E[M(t \wedge b) - M(t \wedge a) \mid \mathcal{F}_a] = 0,$$

which is automatic if $t \le a$, and is the martingale property if $t > a$. It follows that

$$E\big[[\Delta_i(t) - \Delta_i(s)][\Delta_j(t) - \Delta_j(s)] \mid \mathcal{F}_{s \vee t_j}\big] = 0,$$

and then that the same is true when the conditioning is with respect to the smaller $\mathcal{F}_s$.

So,

$$E\big[[M^*(t)]^2 - [M^*(s)]^2 \mid \mathcal{F}_s\big] = E\big[[M^*(t) - M^*(s)]^2 \mid \mathcal{F}_s\big] = \sum_{i=1}^{\infty} E\big[[\Delta_i(t) - \Delta_i(s)]^2 \mid \mathcal{F}_s\big].$$

Therefore, to show that $[M^*(t)]^2 - A^*(t)$ is a martingale, it suffices to check that

$$E\big[[\Delta(t) - \Delta(s)]^2 \mid \mathcal{F}_s\big] = E\big[Y^2[A(t \wedge b) - A(s \vee a)]^+ \mid \mathcal{F}_s\big].$$

Again, the main case is $a \leq s < t \leq b$, in which case, this becomes

$$Y^2 E\big[[M(t) - M(s)]^2 \mid \mathcal{F}_s\big] = Y^2 E\big[A(t) - A(s) \mid \mathcal{F}_s\big],$$

which follows from

$$E[M^2(t) - M^2(s) \mid \mathcal{F}_s] = E[A(t) - A(s) \mid \mathcal{F}_s].$$

But, this is just the statement that $M^2(t) - A(t)$ is a martingale.

Page 202. In Definition 5.12, ... are left continuous with right limits and for which ...

Page 202. Following Definition 5.12, insert: Adapted left continuous processes are jointly measurable – see the proof of Proposition 1.34, or of Proposition 5.13 below. Therefore, the integral defining $||Y||_T^2$ on $[0, T] \times \Omega$ is meaningful, and can be evaluated as an iterated integral. Alternatively, one can think of this as a double integral with respect to a measure on the product space – see Section III.2 of J. Neveu's book "Mathematical Foundations of the Calculus of Probability", for example. This allows the application of theorems such as bounded and dominated convergence to be applied either on the coordinate spaces, or on the product space. The resulting $|| \cdot ||_T$ is not a norm, unless the usual equivalence classes are used. However, it does satisfy the triangle inequality, since it is the usual L_2 norm on the product space.

Page 206. Let the existing Exercise 5.25 be part (a), and add a part (b): (b) Generalize the statement in (a) to the situation in which different integrators are used in the definition of M_1 and M_2. (This proof is harder than the one for part (a). Prove the result first for predictable step functions Y_i, and then pass to the limit.)

Page 208, line -10. There is an extra parenthesis at the end of the display.

Page 211. Insert a new exercise between Exercises 5.36 and 5.37: If $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions, then $B_1(t)B_2(t)$ is a martingale by Exercise 1.100. Find its variance process.

Page 213, middle of the page. ... =0 by Exercise 5.25. Note that ...

Page 229. There is an unimportant constant missing in (6.4).

Page 236. There is a P^x missing in the final display.

Page 237, line 5. R^2 should be R^n.

Page 242. In the next to last line of the proof of Theorem 6.41, the 0 should be t.

Page 249. In Theorem A.4(ii), the X_n's should be nonnegative.

Page 265, line 11. h should be α.