Total Positivity and Renewal Theory

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Suppose that $f(k)$ is a probability density on the positive integers, and let $u(k)$ be the corresponding renewal sequence. Kaluza (1928) and de Bruijn and Erdös (1953) proved several results which relate convexity properties of f to convexity properties of u. We first note that these convexity properties can be formulated in terms of the total positivity of certain orders of the functions $f(i+j+1)$ and $u(i+j)$. This observation permits us to prove an infinite collection of implications which contain the Kaluza and de Bruijn and Erdös results as special cases. In our second result, we show how the imposition of a mild total positivity assumption on $f(k)$ permits one to give a straightforward proof of the fact that $u(n) - u(n+1)$ is asymptotic to a constant multiple of the tail probabilities of f. Continuous time versions of these results are discussed briefly. This work was motivated by a problem in interacting particle systems.

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1. INTRODUCTION

Samuel Karlin has made fundamental contributions to the two fields of total positivity and renewal theory. In this paper, we will investigate some connections between these two areas. We begin by recalling some basic definitions. The equation

$$u(n) = \sum_{k=1}^{n} f(k)u(n-k), \quad n \geq 1, \quad u(0) = 1 \tag{1.1}$$

gives a one-to-one mapping between sequences $\{f(k), k \geq 1\}$ and $\{u(k), k \geq 0\}$ of real numbers with $u(0) = 1$. When $f(k)$ is a probability density on the positive integers, (1.1) is known as the renewal equation. In this case, we can let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables with distribution given by $P(X_i = k) = f(k)$, and set $S_n = X_1 + \cdots + X_n$ and $S_0 = 0$. Then $u(n)$ has the following probabilistic interpretation:

$$u(n) = \sum_{m=0}^{\infty} P(S_m = n) \quad \text{for} \quad n \geq 0. \tag{1.2}$$

The basic renewal theorem states under a mild aperiodicity assumption that

$$u(\infty) = \lim_{n\to\infty} u(n) = \left[\sum_{k=1}^{\infty} kf(k)\right]^{-1}. \tag{1.3}$$

A matrix $M = (m_{i,j})$ is said to be totally positive of order $r \geq 1$ (TP_r) if for every $k \leq r$, every k by k submatrix of M has a determinant which is nonnegative. If these determinants are all strictly positive, the matrix M is said to be strictly totally positive of order r (STP_r). Given a function $g(n)$ on the nonnegative integers, one can consider total positivity properties of the matrix with entries $m_{i,j} = g(i+j)$. Samuel Karlin has been the primary developer of the extensive theory of total positivity, and has demonstrated its usefulness in differential equations, probability theory, and many other parts of mathematics – see his book Karlin (1968), for example.

Karlin (1964) discusses applications of total positivity to Markov chains, and hence indirectly to renewal theory, insofar as renewal theory occurs in their study. There have been a number of investigations of relations between monotonicity and renewal theory – see Brown (1980) and the references there, for example. Little if anything, however, has been done on direct connections between total positivity and renewal theory. We hope to fill this gap to some extent in this paper.

About twenty years ago, when I began to work with Sam Karlin as a graduate student at Stanford, he was heavily involved in total positivity, and encouraged me to work in that area. I read his book, which was then in galley form, but ended up working in pure probability theory instead. I did find one of the Karlin-McGregor (1959) theorems very useful in one paper which I wrote shortly after my thesis – Liggett (1970). It is the rather striking assertion that for a one dimensional continuous time stochastic process, continuity of paths is equivalent to the total positivity of the transition probabilities of the process in the spatial variables. With that exception, this is my first paper related to total positivity. I hope that it partially makes up for my not having followed some of Sam's advice twenty years ago.

The starting point for our first result is the following collection of four facts, which at first glance seem to be unrelated, concerning sequences $f(k)$ and $u(k)$ which are connected by (1.1):

(a) If $f(k) \geq 0$ for all $k \geq 1$, then $u(k) \geq 0$ for all $k \geq 0$.

(b) If $f(k-1)f(k+1) \geq f^2(k)$ for all $k \geq 2$ then $u(k-1)u(k+1) \geq u^2(k)$ for all $k \geq 1$.

(c) If $u(k-1)u(k+1) \geq u^2(k)$ for all $k \geq 1$, then $f(k) \geq 0$ for all $k \geq 1$.

(d) $f(k) = \int_0^\infty x^k d\mu$ for some measure μ and all $k \geq 1$ if and only if $u(k) = \int_0^\infty x^k d\nu$ for some measure ν and all $k \geq 0$.

The first fact is obvious, the second was proved by de Bruijn and Erdös (1953) and the last two were proved by Kaluza (1928) – see also Horn (1970) and Shanbhag (1977).

In order to interpret these facts in terms of total positivity, note that a function $g(k)$ on the nonnegative integers is nonnegative if and only if $g(i+j)$ is TP_1, and that it satisfies $g(k-1)g(k+1) \geq g^2(k)$ for all $k \geq 1$ if and only if $g(i+j)$ is TP_2. Furthermore, g is a moment sequence if and only if $g(i+j)$ is TP_r for all $r \geq 1$ (see Section 7 of Chapter 2 of Karlin (1968) for a continuous version of this result, or obtain the discrete version directly by combining Theorem 1.3 of Shohat and Tamarkin (1943) with the comments on page 18 of Karlin (1968)). Therefore the four facts above are special cases of the following result. Its proof is based on a set of identities which relate determinants based on u to determinants based on f. It will be carried out in the next section.

THEOREM 1. *Suppose that the sequences $f(k)$ and $u(k)$ are connected by (1.1), and let r be a positive integer.*

(a) *If $f(i+j+1)$ is TP_r, then $u(i+j)$ is TP_r.*
(b) *If $u(i+j)$ is TP_{r+1}, then $f(i+j+1)$ is TP_r.*

There are some connections between Theorem 1 and some of the results proved in Karlin (1964). In that paper, Karlin proves that if X_n is a discrete time Markov chain on $\{0,1,\dots\}$ whose transition probabilities are TP_r in the spatial variables, then $u(n+m)=P^0(X_{n+m}=0)$ is TP_r in $n,m\geq 0$ (Theorem 2.7) and $f(n+m)=P^0(\tau_0=n+m)$ is TP_r in $n\geq 0$ and $m\geq 1$, where τ_0 is the hitting time of 0 (Proposition 10.4). By our Theorem 1, the second of these results implies the first.

Over the years, many theorems have been proved which assert that under some assumptions, and in one sense or another, $u(n)-u(n+1)$ behaves like the tail

$$F(n)=\sum_{k=n}^{\infty} f(k)$$

of the probability density f. Karlin (1955) proved an early result in this direction: If r is a positive integer and

$$\sum_{n=1}^{\infty} n^{r+1} f(n)<\infty,$$

then

$$\sum_{n=1}^{\infty} n^{r-1}|u(n)-u(\infty)|<\infty.$$

Other results along these lines are given by Stone (1965) and by Grübel (1982, 1983) – see also the references in the latter papers. Using results of Chover, Ney and Wainger (1973), Embrechts and Omey (1984) proved that

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{F(k)F(n-k)}{F(n)}=2\sum_{k=1}^{\infty} kf(k)<\infty$$

implies that

$$\lim_{n\to\infty} \frac{u(n) - u(n+1)}{F(n+2)} = \left[\sum_{k=1}^{\infty} k f(k)\right]^{-2}.$$

In our second theorem, we show that by imposing a mild total positivity assumption on F, one can give a relatively simple proof of a similar result. Our proof is essentially the same as one of the usual proofs of the ordinary renewal theorem (1.3), but it yields the more refined statement about convergence of ratios.

THEOREM 2. *Suppose that $f(k)$ is a probability density on the positive integers. Assume that $F(i+j+1)$ is TP_2 and that $F(2) > 0$. Then*

$$0 \le u(n) - u(n+1) \le F(n+2). \tag{1.4}$$

Now put

$$\rho = \lim_{n\to\infty} \frac{F(n+1)}{F(n)} \le 1, \tag{1.5}$$

where the limit exists and is positive by monotonicity since $F(i+j+1)$ is TP_2, and $\rho \le 1$ since $F(n)$ is decreasing. Then

$$\lim_{n\to\infty} \frac{u(n) - u(n+1)}{F(n+2)} = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} F(n)\rho^{-n} = \infty \tag{1.6}$$

and

$$\lim_{n\to\infty} \frac{u(n) - u(n+1)}{F(n+2)} = \left[\sum_{k=0}^{\infty} F(k+1)\rho^{-k}\right]^{-2} > 0 \quad \text{if} \quad \sum_{n=1}^{\infty} \frac{F^2(n)}{F(2n)} < \infty. \tag{1.7}$$

REMARKS. (a) The property that $F(i+j+1)$ is TP_2 is also known as DFR (decreasing failure rate). It is slightly weaker than the property that $f(i+j+1)$ is TP_2.

(b) It would be interesting to determine the rate of convergence to zero in (1.6).

EXAMPLES. (a) Fix $0 < \rho \leq 1$, and define $f(k)$ by

$$F(n+1) = \frac{(2n)!}{n!(n+1)!}\left(\frac{\rho}{4}\right)^n \quad \text{for} \quad n \geq 0.$$

Using identity (1.20) in Chapter VI of Liggett (1985), it is not hard to check that the corresponding renewal sequence is determined by

$$u(n) - u(n+1) = \frac{\rho}{4}F(n+1),$$

so that the limit of $[u(n) - u(n+1)]/F(n+2)$ is $1/4$ in this case. This is a case in which (1.7) holds, since $F(n)$ is asymptotic to a constant multiple of $n^{-3/2}\rho^n$.

(b) Consider the distribution of one half of the time until the first return to the origin for the simple symmetric random walk on the integers. It has density

$$f(n) = \frac{1}{2n-1}\binom{2n-1}{n}2^{-2n+1}.$$

The corresponding renewal sequence is of course given by

$$u(n) = \binom{2n}{n}2^{-2n}.$$

Both $u(n) - u(n+1)$ and $f(n)$ are asymptotic to a constant multiple of $n^{-3/2}$, so this provides an example of (1.6).

Continuous time analogues of Theorems 1 and 2 will be discussed briefly in Section 4. We will conclude this section by describing two applications of Theorems 1 and 2 which played important roles in my recent solution in Liggett (1989) of a problem involving exponential rates of convergence for certain interacting particle systems which are known as nearest particle systems. (See Chapter VII of Liggett (1985) for more about these processes.) The first makes precise the statement that if $f(n)$ has exponential tails, then $u(n)$ converges to its limit exponentially rapidly. For our application to interacting particle systems, it is not sufficient to say that $u(n) - u(\infty)$ is exponentially small – the following stronger statement in terms of ratios is needed.

COROLLARY. *Suppose that f is a probability density on the positive integers which satisfies $f(1) < 1$ and is not a geometric density. If either*

(a) $f(i+j+1)$ *is* TP_3, *or*

(b) $F(i+j+1)$ *is* TP_2 *and* $\displaystyle\sum_{n=1}^{\infty}\frac{F^2(n)}{F(2n)} < \infty,$

then

$$\limsup_{n\to\infty} \frac{u(n)-u(\infty)}{u(n)-u(n+1)} \le \frac{1}{1-\rho}, \tag{1.8}$$

where ρ *is defined in* (1.5).

REMARK. In Liggett (1989), we used the obvious fact that (1.8) holds under the stronger assumption that $f(k)$ (and hence by Kaluza's result, also $u(k)$) is a moment sequence. We can now replace the moment sequence assumption in Theorem 1.5 of that paper by the assumption of the above corollary, at the expense of changing the constants somewhat.

PROOF. We can assume that $\rho < 1$, since otherwise (1.8) is trivial. Suppose first that (a) holds. By Theorem 1, $u(i+j)$ is also TP_3. Therefore, after performing two column operations, we see that for $n \ge 0$ and $k \ge 1$, we have

$$\begin{vmatrix} u(n)-u(n+1) & u(n+1)-u(n+2) & u(n+2) \\ u(n+1)-u(n+2) & u(n+2)-u(n+3) & u(n+3) \\ u(n+k)-u(n+k+1) & u(n+k+1)-u(n+k+2) & u(n+k+2) \end{vmatrix} \ge 0.$$

Letting k tend to ∞, it follows that $\Delta(n) = u(n) - u(n+1)$ has the property that $\Delta(i+j)$ is TP_2. Therefore, either $\Delta(n) > 0$ for all $n \ge 0$ or $\Delta(n) = 0$ for all $n \ge 1$. We have excluded the geometric case, so the former case holds. It then follows by monotonicity that

$$\lim_{n\to\infty} \frac{\Delta(n+1)}{\Delta(n)} = \gamma$$

exists. By (1.4), $\gamma \le \rho$. Now write

$$\frac{u(n)-u(\infty)}{u(n)-u(n+1)} = \sum_{k=0}^{\infty} \frac{\Delta(n+k)}{\Delta(n)}$$

and use the dominated convergence theorem to get (1.8). Now assume that (b) holds. Then the limit of $[u(n)-u(n+1)]/F(n+2)$ exists and is strictly positive by Theorem 2, so that (1.8) will follow from the inequality

$$\sum_{k=n}^{\infty} F(k) \le \frac{F(n)}{1-\rho} \quad \text{for} \quad n \ge 1.$$

But this is a consequence of $F(k+1) \leq \rho\, F(k)$, which in turn follows from the monotonicity in the convergence in (1.5).

Our second application of total positivity in renewal theory provides a representation of a stationary renewal process in terms of a sequence of independent and identically distributed random variables. Suppose that $u(0) = 1$, $u(i+j)$ is TP_2, and $0 < u(\infty) < \infty$. By Theorem 1, the corresponding $f(k)$ is a probability density with finite mean. Define a probability measure on the nonnegative integers by $\pi(0) = u(1)$ and

$$\pi(n) = \frac{u(n+1)}{u(n)} - \frac{u(n)}{u(n-1)} \text{for} \quad n \geq 1,$$

which is nonnegative by the total positivity assumption. Let $\{X_i, -\infty < i < \infty\}$ be independent random variables with $P(X_i = n) = \pi(n)$. Now let $\{\eta_i, -\infty < i < \infty\}$ be Bernoulli random variables defined in terms of the X_i's by

$$\eta_i = 1 \quad \text{if and only if} \quad X_{i+k} \leq k \quad \text{for all} \quad k \geq 0.$$

In Theorem 4.6 of Liggett (1989), we proved that $\{\eta_i, -\infty < i < \infty\}$ has the distribution of the set of renewal times of a stationary renewal process whose interarrival times have density $f(k)$. This construction of a stationary renewal process has a number of applications which are discussed in that paper. Of course, for any stationary renewal process, the interarrival times form an i.i.d. sequence. However, the renewal process cannot be expressed directly as a function of this sequence because of the difficulty in initializing the process. Our construction is therefore often quite useful, even though it can only be used if $u(i+j)$ is TP_2.

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2. THE TOTAL POSITIVITY CONNECTION

This section is devoted to the proof of Theorem 1. The main work is in finding and checking a set of identities which relate determinants based on f

with determinants based on u. They are given in the following proposition. We will use the following notation for determinants of matrices constructed from a sequence $g(\cdot)$:

$$g(i_0, \ldots, i_k) = \begin{vmatrix} g(i_0) & g(i_0+1)\cdots g(i_0+k) \\ g(i_1) & g(i_1+1)\cdots g(i_1+k) \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ g(i_k) & g(i_k+1)\cdots g(i_k+k) \end{vmatrix}.$$

PROPOSITION 2.1. *Suppose that the sequences f and u are related by* (1.1). *The following four identities are valid for $n \geq 1$ and $k \geq 1$:*

$$(a) f(n+1, n+2, \ldots, n+k)u(n, n+1, \ldots, n+k) = \sum_{j=1}^{n} f(j, n+1, n+2, \ldots, n+k)u(n-j, n, n+1, \ldots, n+k-1).$$

$$(b) f(n+1, n+2, \ldots, n+k)u(n, n+1, \ldots, n+k-1) = \sum_{j=1}^{n} f(j, n+1, n+2, \ldots, n+k-1)u(n-j, n, n+1, \ldots, n+k-1).$$

(c) $$u(1, 2, \ldots, k) = f(1, 2, \ldots, k).$$

(d) $$u(0, 1, \ldots, k) = f(2, 3, \ldots, k+1).$$

REMARK. Note that identities (a) and (b) bear a resemblance to (1.1). Special cases of the identities in Proposition 2.1 have appeared earlier. Identities (c) and (d) are given in equation (32) in Kaluza (1928). Identity (b) for $k = 1$ is equation (15) in Kaluza (1928). Identity (a) for $k = 1$ is equation (7) in de Bruijn and Erdös (1953).

PROOF. First we show that (c) and (d) follow from (a) and (b). In doing so, we may assume that $f(1, 2, \ldots, k) \neq 0$ and $f(2, 3, \ldots, k+1) \neq 0$ for all $k \geq 1$, since the sequences satisfying these inequalities are dense in the collection of all sequences. Identities (a) and (b) with $n = 1$ give respectively

$$f(2,\dots,k+1)u(1,\dots,k+1) = f(1,\dots,k+1)u(0,\dots,k)$$

and

$$f(2,\dots,k+1)u(1,\dots,k) = f(1,\dots,k)u(0,\dots,k)$$

for $k \geq 1$. Therefore

$$\frac{u(1,\dots,k+1)}{f(1,\dots,k+1)} = \frac{u(0,\dots,k)}{f(2,\dots,k+1)} = \frac{u(1,\dots,k)}{f(1,\dots,k)}$$

for $k \geq 1$. It follows that the expression above is independent of k, and is hence equal to one since $u(1) = f(1)$. This gives both (c) and (d). Turning now to the proof of (a), let $f_i = f(n+1,\dots,n+i,n+i+2,\dots,n+k+1)$ and $u_l = u(n,\dots,n+l-1,n+l+1,\dots,n+k)$ for $0 \leq i,\ l \leq k$, and expand the determinants on the right side of (a) along their top rows to get

$$\text{RHS}(a) = \sum_{j=1}^{n}\sum_{i=0}^{k}\sum_{l=0}^{k} f(j+i)u(n-j+l)(-1)^{i+l} f_i u_i.$$

Now carry out the sum on j using (1.1) to get

$$\text{(2.1)}\qquad \text{RHS}(a) = \sum_{i,l=0}^{k} f_i u_l (-1)^{i+l}\Bigg[u(n+i+l) - \sum_{j=1}^{i} f(j)u(n+i+l-j) - \sum_{j=i+n+1}^{i+n+l} f(j)u(n+i+l-j)\Bigg]$$

$$= \sum_{i,l=0}^{k} f_i u_l (-1)^{i+l}\Bigg[u(n+i+l) - \sum_{j=1}^{i} f(j)u(n+i+l-j) - \sum_{j=0}^{l-1} f(n+i+l-j)u(j)\Bigg].$$

Next we examine separately the sums corresponding to each of the three terms in brackets:

$$\text{(2.2)}\quad \sum_{l=0}^{k} u_l(-1)^l u(n+i+l-j) = u(n+i-j,n,n+1,\dots,n+k-1) = 0$$

for $0 \leq i-j \leq k-1$, where the first equality comes from expanding the determinant on the right along its top row, and the second comes from the fact that the determinant of a matrix with two equal rows is zero. Similarly,

$$\sum_{i=0}^{k} f_i(-1)^i f(n+i+l-j) = f(n+l-j, n, n+1, \dots, n+k-1) = 0 \tag{2.3}$$

for $0 \leq l-j \leq k-1$. Finally,

$$\begin{aligned} \sum_{l=0}^{k} u_l(-1)^l u(n+k+l) &= u(n+k, n, n+1, \dots, n+k-1) \\ &= (-1)^k u(n, \dots, n+k). \end{aligned} \tag{2.4}$$

Using (2.2), (2.3), and (2.4) in (2.1) gives (a). The proof of (b) is similar. This time, let $f_i = f(n+1, \dots, n+i, n+i+2, \dots, n+k)$ and $u_l = u(n, \dots, n+l-1, n+l+1, \dots, n+k)$, and compute

$$\text{RHS}(b) = \sum_{j=1}^{n} \sum_{i=0}^{k-1} \sum_{l=0}^{k} f(j+i)u(n-j+l)(-1)^{i+l} f_i u_l.$$

Using (1.1) again gives

$$\begin{aligned} \text{RHS}(b) = \sum_{i=0}^{k-1} \sum_{l=0}^{k} &(-1)^{i+l} f_i u_l \\ &\times \left[u(n+i+l) - \sum_{j=1}^{i} f(j)u(n+i+l-j) \right. \\ &\left. - \sum_{j=0}^{l-1} f(n+i+l-j)u(j) \right]. \end{aligned} \tag{2.5}$$

To prove (b), use (2.2) and the following identity in (2.5):

$$\sum_{i=0}^{k-1} f_i(-1)^i f(n+i+l-j) = f(n+l-j, n+1, \dots, n+k-1)$$

$$= \begin{cases} 0 & \text{if } \ 1 \leq l-j \leq k-1 \\ (-1)^{k+1} f(n+1, \dots, n+k) & \text{if } \ l-j = k. \end{cases}$$

This completes the proof of the proposition. ∎

The following result is a special case of Theorem 3.3 of Chapter 2 of Karlin (1968).

PROPOSITION 2.2. *The sequence* $g(\cdot)$ *has the property that* $g(i+j)$ *is* STP_r *for* $0 \le i,\ j \le N$ *if and only if* $g(n,\ n+1,\dots,n+k) > 0$ *for all* $n \ge 0$ *and all* $k \ge 0$ *with* $k < r$ *and* $n + 2k \le 2N$.

PROOF OF THEOREM 1. We will prove the slightly modified version of the theorem in which TP is replaced by STP in both the hypotheses and the conclusions. To then remove the S's, it is necessary to approximate totally positive sequences by strictly totally positive sequences. The simplest way to do this is to add a small strictly totally positive sequence to the given sequence. See Chapters 2 and 3 of Karlin (1968) for details. To prove part (a) of the theorem, assume that r is a positive integer and that $f(i+j+1)$ is STP_r for $i,\ j \ge 0$. By Proposition 2.2, in order to show that $u(i+j)$ is STP_r for $i,\ j \ge 0$, it is enough to show that

$$u(n,\dots,n+k) > 0 \tag{2.6}$$

for all $n \ge 0$ and all $0 \le k < r$. This is true for $n = 0$ and $n = 1$ by parts (c) and (d) of Proposition 2.1. We proceed now by induction on n. Suppose that (2.6) is true for $n < N$ and all $0 \le k < r$. Put $n = N$ in part (a) of Proposition 2.1. Then all of the determinants which appear in that identity except $u(N,\dots,N+k)$ are strictly positive by the total positivity assumption on $f(\cdot)$, the induction assumption on $u(\cdot)$, and Proposition 2.2. Therefore, $u(N,\dots,N+k) > 0$ as well. The proof of part (b) of Theorem 1 is similar, using part (b) of Proposition 2.1 in place of part (a) for the induction step. ∎

3. THE CONVERGENCE THEOREM

Here we prove Theorem 2. We will assume throughout that $f(\cdot)$ is a probability density on the positive integers whose tail probabilities $F(\cdot)$

have the property that $F(i+j+1)$ is TP_2. The key step is to find a useful recurrence relation satisfied by

$$v(n) = \frac{u(n) - u(n+1)}{F(n+2)}. \tag{3.1}$$

The first thing one might think of doing is to take differences in (1.1), and then divide by $F(n+2)$. This does lead to a recursion for $v(n)$, but one which is not particularly useful, and which does not take sufficient advantage of the total positivity assumption in Theorem 2.

A better recursion is obtained by coupling together two copies of the renewal process corresponding to the density $f(\cdot)$. To carry out this coupling, let

$$p(k) = \frac{f(k)}{F(k)} \quad \text{for} \quad k \geq 1 \tag{3.2}$$

be the conditional probability of having a renewal at time k given that there has been no renewal prior to that time. Note that

$$\prod_{k=1}^{n} [1 - p(k)] = F(n+1) \tag{3.3}$$

and that the total positivity assumption on $F(\cdot)$ is equivalent to the statement that $p(k)$ is a decreasing function of k. Construct two sequences $\{\eta(n), n \geq 1\}$ and $\{\zeta(n), n \geq 0\}$ of Bernoulli random variables with values 0 and 1 in the following way: $\eta(1) = \zeta(0) = 1$, $P\{\zeta(1) = 1\} = f(1)$, $\zeta(n) \leq \eta(n)$ for all $n \geq 1$, and conditional on $\eta(k)$ and $\zeta(k)$ for $k < n$,

$$\begin{aligned} \eta(n) = \zeta(n) = 1 &\text{ with probability } p(n - \max\{j < n : \zeta(j) = 1\}), \\ \eta(n) = \zeta(n) = 0 &\text{ with probability } 1 - p(n - \max\{j < n : \eta(j) = 1\}), \text{ and} \\ \eta(n) = 1, \zeta(n) = 0 &\text{ with probability } p(n - \max\{j < n : \eta(j) = 1\}) \\ &\qquad - p(n - \max\{j < n : \zeta(j) = 1\}). \end{aligned}$$

This construction is possible because of the monotonicity of $p(k)$. It is easy to check that the marginal distribution of the sequence $\{\eta(n),\ n \geq 1\}$ is that of a sequence of renewals with first renewal at time one and inter-renewal density $f(\cdot)$. Similarly, the marginal distribution of the sequence $\{\zeta(n),\ n \geq 0\}$ is that of a sequence of renewals with first renewal at time zero and inter-renewal density $f(\cdot)$. Therefore, we can use a decomposition according to the time of the last discrepancy between η and ζ to perform the following calculation for $n \geq 1$, keeping in mind that if there is a discrepancy at m, then $\zeta(k) = 0$ for $k \leq m$:

$$\begin{aligned}
u(n) - u(n+1) &= P[\eta(n+1) = 1] - P[\zeta(n+1) = 1] \\
&= P[\eta(n+1) = 1, \zeta(n+1) = 0] \\
&= \sum_{k=1}^{n} P[\eta(n+1) = 1,\ \zeta(n+1) = 0,\ \eta(k) = 1,\ \zeta(k) = 0,\ \eta(j) = 0 \\
&\qquad\qquad\qquad \text{for all } k < j \leq n] \\
&= \sum_{k=1}^{n} P[\eta(k) = 1,\ \zeta(k) = 0]\left\{\prod_{j=1}^{n-k}[1 - p(j)]\right\}[p(n-k+1) - p(n+1)] \\
&= \sum_{k=1}^{n}[u(k-1) - u(k)]F(n-k+1)\left[\frac{F(n+2)}{F(n+1)} - \frac{F(n-k+2)}{F(n-k+1)}\right],
\end{aligned}$$

where the last equality follows from (3.3). Rewriting this in terms of $v(\cdot)$ using (3.1), and then making a change of variable in the summation gives

$$\begin{aligned}
v(n) &= \sum_{k=1}^{n} v(k-1)F(k+1)\left[\frac{F(n-k+1)}{F(n+1)} - \frac{F(n-k+2)}{F(n+2)}\right] \\
&= \sum_{k=1}^{n} g_{n-k}(k)v(n-k),
\end{aligned} \tag{3.4}$$

where

$$g_m(k) = F(m+2)\left[\frac{F(k)}{F(m+k+1)} - \frac{F(k+1)}{F(m+k+2)}\right] \tag{3.5}$$

for $m \geq 0$ and $k \geq 1$. Note that $g_m(k) \geq 0$ by the total positivity assumption, and is a sub probability density:

$$\sum_{k=1}^{\infty} g_m(k) = 1 - F(m+2)\rho^{-m-1} < 1 \tag{3.6}$$

by (1.5). Thus since $v(0) = 1$, (3.4) exhibits $v(n)$ as the renewal sequence corresponding to inter-renewal times which are independent, but whose distributions are time dependent and defective.

Note that if $F(n)$ is replaced by $F(n)s^{n-1}$ for some $0 < s < 1$, then $g_m(k)$, and hence $v(n)$, remains unchanged. This explains why the statement of Theorem 2 does not depend on the "exponential part" of the density $f(n)$. We can now proceed to the proof of Theorem 2.

PROOF OF THEOREM 2. Since $g_m(k)$ is a sub probability density and $v(0) = 1$, (3.4) implies that

$$0 \le v(n) \le 1 \tag{3.7}$$

for all $n \ge 0$. This gives (1.4). Now note that

$$g(k) = \lim_{m\to\infty} g_m(k) = F(k)\rho^{-k+1} - F(k+1)\rho^{-k} \tag{3.8}$$

for $k \ge 1$ by (1.5), and that $g(\cdot)$ sums to one if and only if $\lim_k F(k)\rho^{-k} = 0$. By (3.6), (3.8) and Scheffé's Theorem,

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} |g_m(k) - g(k)| = 0. \tag{3.9}$$

Define a sequence of Bernoulli random variables $\{\eta(n), n \ge 0\}$ by $\eta(0) = 1$ and

$$P[\eta(k+1) = \cdots = \eta(n-1) = 0, \eta(n) = 1 \mid \eta(0), \eta(1), \ldots, \eta(k-1), \eta(k) = 1] = g_k(n-k).$$

Using (3.4) and induction, it follows that $v(n) = P[\eta(n) = 1]$. Then, considering the distances between N and the nearest indices smaller and larger than N at which η is one, we have

$$\begin{aligned} P[\eta(m) = 1 \text{ for some } m > N] &= \sum_{k=0}^{N}\sum_{l=1}^{\infty} v(N-k) g_{N-k}(k+l) \\ &= \sum_{k=0}^{N} v(N-k)F(N-k+2)\left[\frac{F(k+1)}{F(N+2)} - \rho^{-N+k-1}\right] \end{aligned} \tag{3.10}$$

by (3.5) and (1.5). Now consider a subsequence of the N's along which $v(N+n)$ converges for each $-\infty < n < \infty$, say to $w(n)$. Every sequence has such a subsequence by (3.7). Then we can replace n by $N+n$ in (3.4) and pass to the limit along that subsequence using (3.9) to obtain

$$w(n) = \sum_{k=1}^{\infty} g(k)w(n-k) \quad \text{for} \quad -\infty < n < \infty. \tag{3.11}$$

If the sum of $g(\cdot)$ is strictly less than one, then $w(n) = 0$ for all n, so we can conclude that $\lim_n v(n) = 0$. Therefore, we may assume from now on that $g(\cdot)$ sums to one, and hence that $\lim_k F(k)\rho^{-k} = 0$. In this case, (3.11) says that $w(\cdot)$ is a (bounded) harmonic function for a random walk, and hence is a constant C by the Choquet-Deny Theorem. Now pass to the limit in (3.10) along our subsequence using Fatou's Lemma to conclude that

$$C\sum_{k=0}^{\infty} F(k+1)\rho^{-k} \leq 1.$$

If the sum above is infinite, it follows that $C = 0$, and hence that $\lim_n v(n) = 0$. This proves (1.6). Proceeding to the proof of (1.7), we may assume that

$$\sum_{n=1}^{\infty} \frac{F^2(n)}{F(2n)} < \infty. \tag{3.12}$$

The TP_2 assumption on the tail probabilities implies that

$$\frac{F(k+1)F(N-k+2)}{F(N+2)} \leq \frac{F^2(k)}{F(2k)}$$

if $2k \leq N+2$. Therefore (3.12) and the dominated convergence theorem imply that

$$\lim_{N\to\infty} \sum_{k=0}^{N/2} v(N-k)F(N-k+2)\left[\frac{F(k+1)}{F(N+2)} - \rho^{-N+k-1}\right] = C\sum_{k=0}^{\infty} F(k+1)\rho^{-k}.$$

After making the change of variable $l = N-k$, the same argument applied to the other half of the terms in the sum on the right of (3.10) gives

$$\lim_{N\to\infty} \sum_{l=0}^{N/2} v(l)F(l+2)\left[\frac{F(N-l+1)}{F(N+2)} - \rho^{-l-1}\right] = 0.$$

So, we can pass to the limit along the subsequence of N's in (3.10) to conclude that

$$P[\eta(m) = 1 \text{ for infinitely many } m] = C\sum_{k=0}^{\infty} F(k+1)\rho^{-k}. \tag{3.13}$$

This determines the value of C, so it follows that the limit of $v(n)$ exists along the full sequence, and

$$\lim_{n\to\infty} v(n) = \frac{P[n(m) = 1 \text{ for infinitely many } m]}{\sum_{k=0}^{\infty} F(k+1)\rho^{-k}}. \tag{3.14}$$

It remains to show that

$$P[\eta(m) = 1 \text{ for infinitely many } m] = \left[\sum_{k=0}^{\infty} F(k+1)\rho^{-k}\right]^{-1}. \tag{3.15}$$

To do so, let

$$z(n) = P[\eta(m) = 1 \text{ for some } m \geq n]. \tag{3.16}$$

The idea is to show that $z(n)$ is the renewal sequence associated with the density $g(k)$. Once this is done, we can let n tend to ∞ in (3.16) using the ordinary renewal theorem to obtain (3.15). To see that $z(n)$ satisfies the renewal equation for g, write for $n \geq 1$:

$$\begin{aligned}
&\sum_{k=1}^{n} g(k)z(n-k) - z(n) \\
&\quad = \sum_{k=1}^{n} F(k)\rho^{-k+1}[z(n-k) - z(n-k+1)] - F(n+1)\rho^{-n} \\
&\quad = \sum_{k=1}^{n} F(k)\rho^{-k+1}P[\eta(n-k) = 1,\ \eta(j) = 0 \text{ for all } j > n-k] \\
&\quad\quad - F(n+2)\rho^{-n} \\
&\quad = \sum_{k=1}^{n} F(k)\rho^{-k+1}v(n-k)F(n-k+2)\rho^{-(n-k+1)} - F(n+1)\rho^{-n} \\
&\quad = \rho^{-n}\left\{\sum_{k=1}^{n} F(k)[u(n-k) - u(n-k+1)] - F(n+1)\right\} \\
&\quad = \rho^{-n}\left\{\sum_{k=1}^{n} u(n-k)f(k) - u(n)\right\} = 0.
\end{aligned}$$

In the above computation, we have summed by parts in the first and next to last steps, and have used the renewal equation (1.1) in the last step. The third equality follows from the construction of the sequence $\eta(n)$ and (3.6), and the fourth comes from the definition of $v(n)$ in (3.1). ∎

4. CONTINUOUS TIME

In this section, we will give a brief description of the continuous time analogues of the results which were presented earlier for discrete time. Let $f(t)$ be a continuous probability density on $[0,\infty)$, and let $u(t)$ be the corresponding renewal density, i.e., the sum for $k \geq 1$ of the k-fold convolutions of f with itself. Then $u(t)$ satisfies the renewal equation

$$u(t) = f(t) + \int_0^t f(s)u(t-s)\,ds, \quad t \geq 0. \tag{4.1}$$

The analogue of Theorem 1 is

THEOREM 3. *Let r be a positive integer.*

(a) *If* $f(s+t)$ *is* TP_r, *then* $u(s+t)$ *is* TP_r.
(b) *If* $u(s+t)$ *is* TP_{r+1}, *then* $f(s+t)$ *is* TP_r.

The proof of Theorem 3 is most easily carried out by discretizing $f(\cdot)$, applying Theorem 1 to the discretization, and then passing to the limit. There are continuous time analogues of the identities in Proposition 2.1, but they are not particularly useful for proving Theorem 3. Still, it is probably worth stating some of these analogues. To do so, we need some notation. If $g(\cdot)$ is a function on $[0,\infty)$ which has k continuous derivatives, let

$$g_{k+1}(t_0,\ldots,t_k) = \begin{vmatrix} g(t_0) & g'(t_0)\cdots g^{(k)}(t_0) \\ g(t_1) & g'(t_1)\cdots g^{(k)}(t_1) \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ g(t_k) & g'(t_k)\cdots g^{(k)}(t_k) \end{vmatrix}$$

for $0 \leq t_0 < t_1 < \cdots < t_k$. When there are coincidences among two or more of the t_i's, successive rows are differentiated, so that for example

$$g_3(s,t,t) = \begin{vmatrix} g(s) & g'(s) & g''(s) \\ g(t) & g'(t) & g''(t) \\ g'(t) & g''(t) & g'''(t) \end{vmatrix}.$$

The continuous time analogues of identities (a) and (c) of Proposition 2.1 then become for $k \geq 1$

$$f_k(t,\ldots,t)u_{k+1}(t,\ldots,t) = f_{k+1}(t,\ldots,t)u_k(t,\ldots,t) + \int_0^t f_{k+1}(s,t,\ldots,t)u_{k+1}(t-s,t,\ldots,t)\, ds$$

and

$$u_k(0,\ldots,0) = f_k(0,\ldots,0)$$

respectively. The proof of the first of these is the same as the proof in discrete time – one simply expands the determinants in the integral along the first row, and then uses (4.1). To prove the second identity, simply put $t = 0$ in the first, and then use induction on k. The analogues of (b) and (d) are more complicated, so they will be omitted.

In order to state the continuous time version of Theorem 2, let

$$F(t) = \int_t^\infty f(s)\, ds$$

be the tail probabilities corresponding to the density $f(\cdot)$. Note that $F(s+t)$ is TP_2 in $s,t \geq 0$ if and only if $f(t)/F(t)$ is decreasing in t, so that in this case, we may define

$$\lambda = \lim_{t\to\infty} \frac{f(t)}{F(t)} \geq 0.$$

THEOREM 4. *Suppose that f is a probability density on $[0,\infty)$ such that the tail probabilities satisfy $F(s+t)$ is TP_2 in $s,t \geq 0$. Then the renewal density $u(t)$ is decreasing on $[0,\infty)$, so that we may define a measure ν on $[0,\infty)$ by*

$$d\nu(t) = -\frac{1}{F(t)}\, du(t).$$

$T \geq 0$, *let* ν_T *be the measure on* $[-T,\infty)$ *which is obtained by translat-* *to the left by a distance* T. *Then* ν_T *converges vaguely as* T *tends to* *ity to the zero measure if*

$$\int_0^\infty F(t)e^{\lambda t}\, dt = \infty,$$

converges vaguely to a strictly positive multiple of Lebesgue measure if

$$\int_0^\infty \frac{F^2(t)}{F(2t)}\, dt < \infty.$$

he proof of Theorem 4 is similar to the proof of Theorem 2 which is given ection 3. We will simply write down the continuous time analogues of more important formulas used there. The continuous time versions of) and (3.5) are

$$d\nu(t) = -d\frac{f(t)}{F(t)} + \int_0^t g_s(t-s)\, d\nu(s)\, dt$$

ere the first term on the right is the (finite) measure obtained by differ- iating the increasing function $-f(t)/F(t)$, and

$$g_s(t) = -F(s)\frac{d}{dt}\frac{F(t)}{F(t+s)}$$

pectively. Note that the TP_2 assumption on $F(\cdot)$ implies again that g_s a sub probability density on $[0,\infty)$, and that

$$\lim_{s\to\infty} g_s(t) = f(t)e^{\lambda t} - \lambda F(t)e^{\lambda t}.$$

rthermore,

$$\int_0^T g_s(t)\, dt = 1 - \frac{F(s)F(T)}{F(T+s)},$$

nich is increasing in s for each T, so that the measures with densities g_s e stochastically decreasing in s. This can be used to show that $\nu(T+[a,b])$ uniformly bounded in T for fixed $a < b$. Therefore the proof can be arried out as before.

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