The Asympotic Shapley Value for a Simple Market Game

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Abstract

We consider the game in which b buyers seek to purchase 1 unit of an indivisible good from s sellers, each of whom have k units to sell. The good is worth 0 to each seller and 1 to each buyer. Using results from Brownian motion, we find a closed form solution for the limiting Shapley value as s and b increase without bound. This asymptotic value depends upon the store size k, the limiting ratio b/ks of buyers to items for sale, and the limiting ratio $[ks - b]/\sqrt{b+s}$ of the excess supply relative to the square root of the number of market participants.

Keywords Shapley Value; Core; Many Players

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1 Introduction

An important issue in economics is the extent to which vigorous competition causes firm profits to vanish. Adam Smith explained why increasing the demand for a good would lead to a higher price, but, unlike Edgeworth, he didn't envisage situations in which a tiny reduction in demand would induce the price to collapse and cause the sellers' surplus to vanish.[1] Yet this pathology is the case in the simple many-player market game examined in this note.

In this note we consider a simple market game with many symmetric buyers and sellers. In this market game, both the core and the Nash equilibrium in prices (Bertrand competition) implicitly entail competition so vigorous that they can result in the total surplus on one side of the market falling to zero when there is a slight increase in the number of players on that side of the market. We show that the Shapley value leads to a natural continuous interpolation between the two extreme situations, and is thereby the preferred solution concept.

The market game finds each of s sellers possessing k units of an indivisible good to sell, and the good is worth 0 to each seller. There are b buyers each seeking to purchase 1 unit, and the good is worth 1 to each buyer. Obviously, k, s, and b are positive integers, and this is the simplest game that captures trade in a market with many agents. When $k > 1$, each seller is a "store" with k units for sale.

When buyers exceed the number of units for sale ($b > ks$), the core of this game has a transaction price of 1; the Nash equilibrium in prices (Bertrand competition) also predicts a price of 1. When the number of units for sale exceed the number of buyers ($ks > b$), these equilibrium concepts predict a transaction price of 0. On the other hand, when $ks = b$, these solution concepts admit any price between 0 and 1.

While these results parallel our ideas of perfect competition, they are not reasonable predictions for the outcome of generalized bargaining situations. For example, when $k = 1$ and s is large, say $s = 10^6$, it is preposterous to predict that s sellers facing $b = s + 1$ buyers would each obtain a price of 1, whilst the addition of two more sellers would drop the price to 0. The Shapley value smooths the discontinuity for this game in a manner consonant with the economic landscape: the addition of buyers induces a gradual increase in the equilibrium price.

Traditionally, the Shapley value has been viewed as more "cooperative" than the core, involving the balancing of probabilities of various coalitons being formed. Recently, there has

[1] At the exact middle of Chapter VII, "Of the Natural and Market Price of Commodities," Adam Smith [5] asserts that "A public mourning raises the price of black cloth ... and augments the profits of the merchants who possess any considerable quantity of it. ... It sinks the price of coloured silks and cloths, and thereby reduces the profits of the merchants who have any considerable quantity of them upon hand."

been considerable interest in non-cooperative rationales for the Shapley value. Two influential examples are Gul's [1] model of random bilaterial contracting and Hart and Mas-Colell's [2] model of sequential bilateral offers.

While the behavior of the Shapley value is clear when there are a small numbers of players, its behavior is not well understood (or known) as the size of the market grows very large. Shapley and Shubik [4] analyze this problem when $k = 1$ and provide asymptotic results for two cases. In the first the ratio of buyers to sellers is fixed with $b = \alpha s$; in the second the difference d between the number of buyers and sellers is a fixed constant d: $b = s + d$. When $\alpha > 1$, the Shapley value converges to 1 and coincides with the core as the size of the market increases. Similarly, the Shapley value converges to 0 when $\alpha < 1$, just like the core. When the difference between the number of buyers and sellers is d, the Shapley value converges to 0.5 as the size of the market increases. In sharp contrast, the core yields equilibrium prices of 1 when $d > 0$, 0 when $d < 0$, and the equilibrium price is indeterminate when $d = 0$.

We show that the asymptotic Shapley value for this market game is not restricted to the values 0, 1/2, and 1; in fact, all values between 0 and 1 are possible. Let $M \equiv b + s$ denote the size of the market, and let $d \equiv ks - b$ denote the excess supply. Using an old result about the Brownian bridge, we provide a closed form expression for the limit of the Shapley value of this game as a function of $u \equiv lim_{b,s\to\infty} d/\sqrt{M}$. As long as the difference d between units for sale and buyers grows in proportion to the square root of the size of the market, the asymptotic Shapley value (for each unit) can take on any intermediate value between 0 and 1.

One way to decide which of several solution concepts is most appropriate for a particular game is to select the concept which produces the most intuitive or plausible answer. On this ground, we find the Shapley value to be more appropriate than either the core or the Nash equilibrium for our simple market bargaining game.

2 The Closed Form Solution

We now present a closed form solution which gives the asymptotic Shapley value as s and b increase to ∞. A particularly interesting aspect of our solution is that for each price p between 0 and 1, there are growth rates for s and b such that the Shapley value converges to p: asymptotically, the Shapley value traces out each possible price. This stands in stark contrast to the other solution concepts which yield prices of 0, 1, or an indeterminate price (or all prices) between 0 and 1.

To begin set $k = 1$, and let $V(b, s)$ denote the Shapley value for a seller in this game (naturally, the value added by all players is $v \equiv min(b, s)$ so that a buyer's Shapley value is

$[v - sV(b,s)]/b)$. To compute $V(b,s)$, consider those permutations of the $b+s$ players in which a given seller is the $i+1^{st}$ player. There is some number j of buyers amongst the first i players in the permutation whence there are $i-j$ sellers amongst the first i players. This seller's value added is 1 if the number j of buyers is strictly greater than the number $i-j$ of sellers: the seller's value added is 1 if and only if $2j > i$. Clearly, the probability that this seller is the $i+1^{st}$ player is $1/(b+s)$. The product $\binom{b}{j}\binom{s-1}{i-j}/\binom{s+b-1}{i}$ of binomial coefficients is the probability that the first i players in a permutation has j buyers and $i-j$ sellers given that the $i+1^{st}$ player is a seller. Consequently, $V(b,s)$ can be written as

$$V(b,s) = \frac{1}{b+s} \sum_{i=0}^{s+b-1} \sum_{2j>i} \frac{\binom{b}{j}\binom{s-1}{i-j}}{\binom{s+b-1}{i}}.$$

More generally, when each seller is a store of integer size k with $k \geq 1$, a similar argument to the one above reveals that the expression for $V(s,b)$, the Shapley value for each seller (who has k units for sale), is

$$V(b,s) = \frac{1}{b+s} \sum_{i=0}^{s+b-1} \sum_{(k+1)j>ik} \min\{k, j-(i-j)k\} \frac{\binom{b}{j}\binom{s-1}{i-j}}{\binom{s+b-1}{i}}.$$

Theorem: Let $b, s \to \infty$ so that $b/ks \to \alpha$.

(a) If $\alpha < 1$, then $V(b,s) \to 0$.

(b) If $\alpha > 1$, then $V(b,s) \to k$.

(c) If $\alpha = 1$, suppose that

$$\frac{ks-b}{\sqrt{b+s}} \to u.$$

Then

$$V(b,s) \to \frac{k^2}{\sqrt{2\pi}} \int_0^\infty \frac{x^2}{u^2+kx^2} e^{-x^2/2} dx \quad \text{if } u \geq 0,$$

and

$$V(b,s) \to k - \frac{k^2}{\sqrt{2\pi}} \int_0^\infty \frac{x^2}{u^2+kx^2} e^{-x^2/2} dx \quad \text{if } u \leq 0.$$

Proof : We first write V in terms of a simple random walk. Take $0 < p < 1$, let $X_1, X_2, \ldots$ be independent identically distributed Bernoulli random variables with

$$P(X_i = 1) = p, \quad P(X_i = 0) = q = 1-p,$$

and let $S_m = X_1 + \cdots + X_m$ be the corresponding partial sums. Setting $N = b+s-1$, we see that

$$P(S_b = j \mid S_N = i) = \frac{P(S_b = j)P(S_{s-1} = i-j)}{P(S_N = i)} = \frac{\binom{b}{j} p^j q^{b-j} \binom{s-1}{i-j} p^{i-j} q^{s-1-i+j}}{\binom{N}{i} p^i q^{N-i}} = \frac{\binom{b}{j}\binom{s-1}{i-j}}{\binom{s+b-1}{i}}.$$

Therefore,

$$\begin{aligned} V(b,s) &= \frac{1}{b+s}\sum_{i=0}^{s+b-1}\sum_{(k+1)j>ik}\min\{k, j-(i-j)k\}P(S_b=j \mid S_N=i) \\ &= \frac{1}{N+1}\sum_{i=0}^{N} E\Big\{\min\{k,(k+1)S_b-ik\};(k+1)S_b>ik \mid S_N=i\Big\}, \end{aligned}$$

where $E(X;A)\equiv E(X\cdot 1_A)$ for a random variable X and a set A.

The process $\{Y_N(t):0\le t\le 1\}$ that appears in standard invariance principles is defined by ($[x]\equiv$ integer part of x):

$$Y_N(t)=\frac{S_{[Nt]}-[Nt]p}{\sqrt{N}}, \quad 0\le t\le 1.$$

Define the random variable $Z_{b,i}$ by $Z_{b,i}=(k+1)S_b-ik$ and the set $A_{N,i}$ by $A_{N,i}=\{S_N=i\}$. Writing $Z_{b,i}$ and $A_{N,i}$ in terms of Y_N, we have $Z_{b,i}=(k+1)\sqrt{N}\,Y_N(\frac{b}{N})+(k+1)bp-ik$ and $A_{N,i}=\{Y_N(1)=\frac{i-Np}{\sqrt{N}}\}$, so that

$$V(b,s) = \frac{1}{N+1}\sum_{i=0}^{N} E\Big\{\min\{k,Z_{b,i}\};Z_{b,i}>0 \mid A_{N,i}\Big\}.$$

Suppose $[ks-b]/\sqrt{b+s} \to u$ with $u\in[-\infty,\infty]$ and $b/(b+s)\to t$. Because $Z_{b,i}$ is integer-valued, we have

$$E[\min\{k,Z_{b,i}\};Z_{b,i}>0 \mid A_{N,i}]=E[Z_{b,i}\,;1\le Z_{b,i}<k \mid A_{N,i}]+kP(Z_{b,i}\ge k \mid A_{N,i}).$$

By [3, Theorem 4], $(Y_N(\frac{b}{N}) \mid A_{N,i}) = (Y_N(\frac{b}{N}) \mid Y_N(1)=(i-Np)/\sqrt{N})$ converges weakly to $Y(t)=\sqrt{p(1-p)}B_0(t)$, where $B_0(t)$ is the standard Brownian bridge.[2] As a consequence, the first term is seen to go to 0 as $N\to\infty$ because $(Y_N(\frac{b}{N}) \mid A_{N,i})$ is approximately normal so that the probability that $Z_{b,i}=(k+1)\sqrt{N}\,Y_N(\frac{b}{N})+(k+1)bp-ik$ lies in a bounded interval goes to zero.

To find the second term, set $i=[(b+s)p]$. Again, using [3, Theorem 4], we have

$$P(Z_{b,i}\ge k \mid A_{N,i})=P(Y_N(\frac{b}{N})\ge\frac{k(i+1)-(k+1)bp}{(k+1)\sqrt{N}} \mid A_{N,i})\to P(Y(t)\ge\frac{up}{k+1}). \quad (1)$$

Using (1) and the bounded convergence theorem, we have

$$V(b,s)\to k\int_0^1 P\Big(\sqrt{p(1-p)}\,B_0(t)>\frac{up}{k+1}\Big)dp \;=\; k\int_0^1 P\Big(B_0(t)>\frac{u}{k+1}\sqrt{\frac{p}{1-p}}\Big)dp. \quad (2)$$

[2]Let $B(t)$ be standard Brownian motion. The Brownian bridge $B_0(t)$ is obtained by "conditioning" $B(\cdot)$ on the zero probability event $\{B(1)=0\}$. Distributionally, it can be written as $B_0(t)=B(t)-tB(1)$.

If $t < k/(k+1)$, then $u = +\infty$, so the right of (2) is zero. If $t > k/(k+1)$, $u = -\infty$, so the right of (2) is 1.

If $t = k/(k+1)$, then u can take any value. We show the result for the case $u \geq 0$ (the case $u \leq 0$ is almost identical). Because $B_0(k/(k+1))$ is $N(0, k/(k+1)^2)$, $B_0(k/(k+1))/[\sqrt{k}/(k+1)]$ is a standard normal random variable, which we denote by Z. Define the function $g(x,p)$ to be 1 if $x \geq u\sqrt{p}/\sqrt{k(1-p)}$ and 0 otherwise. Then utilizing (2) and interchanging the order of integration, we obtain

$$\begin{aligned} V(b,s) &\to k\int_0^1 P\Big(Z > \frac{k+1}{\sqrt{k}} \cdot \frac{u}{k+1}\frac{\sqrt{p}}{\sqrt{1-p}}\Big)dp &&= k\,\frac{1}{\sqrt{2\pi}}\int_0^1 [\int_0^\infty e^{-x^2/2} g(x,p)dx]dp \\ &= k\,\frac{1}{\sqrt{2\pi}}\int_0^\infty [\int_0^1 g(x,p)dp]\, e^{-x^2/2}dx &&= k\,\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{kx^2}{u^2+kx^2}\, e^{-x^2/2}dx. \end{aligned}$$

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The equilibrium market price for each unit of the item is $V(b,s)/k$. In a large market, the closed form solution reveals that when $0 < u < \infty$, $lim_{b,s\to\infty} V(b,s)/k$ is strictly increasing in k, with limit 1/2: when there is an excess supply ($ks > b$), larger stores induce an increase in the equilibrium market price. [The opposite is true when $-\infty < u < 0$.]

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