T. Liggett Mathematics 31B – First Midterm Solutions April 17, 2009

(9) 1. The size of a certain population grows exponentially, and is $P(t)$ at time t. Given that $P(0) = 10$ and $P(1) = 11$, at what time will the size of the population be 30?

Solution. By exponential growth, $P(t) = 10e^{kt}$ for some k. Since $11 = P(1) = 10e^k$, $e^k = 11/10$. The desired t satisfies $30 = 10e^{kt} = 10(11/10)^t$. Therefore, $\ln 3 = t \ln(11/10)$, so that $t = \ln 3/\ln(11/10)$.

(16) 2. Let $f(x) = \frac{x^3}{x^2+1}$.

(a) Show that f is strictly increasing, and hence one-to-one.

Solution.

$$f'(x) = \frac{3x^2 + x^4}{(x^2+1)^2},$$

which is strictly positive, except for $x = 0$. Therefore, f is strictly increasing.

(b) Compute $g'(-\frac{1}{2})$, where g is the inverse of f.

Solution. Since $f(-1) = -\frac{1}{2}$, $g(-\frac{1}{2}) = -1$. Therefore, $g'(-\frac{1}{2}) = 1/f'(-1) = 1$.

(14) 3. (a) Compute

$$\frac{d}{dx} e^{-x}\sqrt{e^{2x}+1} = -\frac{e^{-x}}{\sqrt{1+e^{2x}}}$$

(b) Evaluate

$$\int_0^2 x e^{x^2}\,dx = \frac{e^4 - 1}{2}.$$

(11) 4. Use logarithmic differentiation to compute y', where

$$y = \sqrt{\frac{x(x+2)}{(2x+1)(2x+2)}}$$

Solution.

$$\ln y = \frac{1}{2}(\ln x + \ln(x+2) - \ln(2x+1) - \ln(2x+2)).$$

Therefore,

$$y'/y = \frac{1}{2}\Big(\frac{1}{x} + \frac{1}{x+2} - 2\frac{1}{2x+1} - \frac{1}{x+1}\Big) = \frac{2 + 2x - x^2}{2x(1+x)(x+2)(2x+1)},$$

so that

$$y' = \sqrt{\frac{x(x+2)}{(2x+1)(2x+2)}}\Big[\frac{2+2x-x^2}{2x(1+x)(x+2)(2x+1)}\Big].$$

(15) 5. Evaluate the following limits:

Solution. In both cases, use L'Hopital's rule.

$$(a)\ \lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}=\lim_{t\to 0}\left[\frac{1}{2\sqrt{1-t}}+\frac{1}{2\sqrt{1+t}}\right]=1.$$

$$(b)\ \lim_{x\to 1}x^{1/(1-x)}$$

First compute

$$\lim_{x\to 1}\ln x^{1/(1-x)}=\lim_{x\to 1}\frac{\ln x}{1-x}=\lim_{x\to 1}\frac{1/x}{-1}=-1.$$

Therefore,

$$\lim_{x\to 1}x^{1/(1-x)}=e^{-1}.$$

(10) 6. (a) Solve $\log_3 y+3\log_3 y^2=14$ for y.
Solution. This is equivalent to $\log_3 y^7=14$, so $y=9$.
(b) Evaluate $\sin^{-1}(-1/\sqrt{2})$.
Solution. We want an angle θ so that $-\pi/2\le\theta\le\pi/2$ and $\sin\theta=-1/\sqrt{2}$, so $\theta=-\pi/4$.
(12) 7. Compute

$$\int_{-5}^{5}\frac{1}{x^2+25}dx$$

Solution. Make the substitution $x=5u$, so that $dx=5du$. Then

$$\int_{-5}^{5}\frac{1}{x^2+25}dx=\frac{1}{5}\int_{-1}^{1}\frac{1}{u^2+1}du=\frac{1}{5}\tan^{-1}u\Big|_{-1}^{1}=\frac{1}{5}\left(\frac{\pi}{4}-\frac{-\pi}{4}\right)=\frac{\pi}{10}.$$

(13) 8. Prove the following: If $f(t)$ satisfies $f'(t)=2f(t)$ and $f(0)=3$, then $f(t)=3e^{2t}$.

Proof. Compute

$$\frac{d}{dt}e^{-2t}f(t)=e^{-2t}f'(t)-2e^{-2t}f(t)=e^{-2t}[f'(t)-f(t)]=0.$$

Therefore, $e^{-2t}f(t)$ is constant. Its value at $t=0$ is 3, so the constant is 3. So, $f(t)=3e^{2t}$.