T. Liggett Mathematics 31A – Second Midterm Solutions February 27, 2009

(12) 1. Find the absolute maximum and minimum of $y = x^4 - 2x^2$ on the interval $[-2, 1]$.

Solution. $y' = 4x^3 - 4x = 4x(x-1)(x+1)$, which is zero at $x = -1, 0, 1$. Therefore, the absolute maximum and minimum must occur at $x = -2, -1, 0$ and/or 1. The values of y at these points are 8,-1,0 and -1 respectively. Therefore, the maximum value is 8, and occurs at $x = -2$, while the minimum value is -1, and occurs at $x = \pm 1$.

(13) 2. Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius 1.

Solution. Use the upper half of the unit circle centered at the origin, and let (x, y) be the coordinates of the point in the first quadrant at which the rectangle meets the circle. Then $x^2 + y^2 = 1$ and the area of the rectangle is $A(x) = 2xy = 2x\sqrt{1-x^2}$, so we must maximize $A(x)$ for $0 \le x \le 1$. Computing the derivative gives $A'(x) = 2(1-2x^2)/\sqrt{1-x^2}$. Since $A(0) = A(1) = 0$, the maximum must occur at the interior critical point, $x = 1/\sqrt{2}$. So, the inscribed rectangle of largest area has dimensions $\sqrt{2} \times 1/\sqrt{2}$.

(10) 3. (a) Find

$$\int (\sin x + \cos(3x+2))dx.$$

Solution. $-\cos x + \frac{1}{3}\sin(3x+2) + C$.

(b) Find the function f that satisfies $f'(t) = t^{-3/2}$ and $f(4) = 1$.

Solution. Antidifferentiating gives $f(t) = -2/\sqrt{t} + C$. The initial condition gives $C = 2$, so $f(t) = -2/\sqrt{t} + 2$.

(15) 4. Let $g(x) = x^5 - 5x + 2$.

(a) Find the critical points, and determine in each case whether it is local maximum, local minimum, or neither.

Solution. $g'(x) = 5x^4 - 5$ and $g''(x) = 20x^3$. Therefore the critical points are $x = \pm 1$. $g''(-1) = -20$ and $g''(1) = 20$, so by the second derivative test, -1 is a local maximum and $+1$ is a local minimum.

(b) Find the inflection points and the intervals on which the graph is concave up or concave down.

Solution. Since g'' changes sign only at $x = 0$, this is the only inflection point. g is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

(11) 5. The base x of the right triangle in the figure below increases at rate 5 cm/sec, while the height remains constant at 20 cm. At what rate is the angle θ changing when $x = 20$ cm?

Solution. The variables θ and x are related by

$$\tan\theta = 20/x,$$

so by the chain rule,

$$\sec^2\theta\frac{d\theta}{dt} = -\frac{20}{x^2}\frac{dx}{dt}.$$

When $x = 20$, $\theta = \pi/4$ and $\sec^2\theta = 2$. Therefore,

$$\frac{d\theta}{dt} = -\frac{1}{8}\ rad/sec.$$

(14) 6. Given that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad and \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

evaluate

$$(a)\ \sum_{j=3}^{50} j(j-1) = \sum_{j=1}^{50} j(j-1) - 2 = \frac{(50)(51)(101)}{6} - \frac{(50)(51)}{2} - 2 = 41640.$$

$$(b)\ \lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2} = \lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2} = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}.$$

(10) 7. (a) Compute the Riemann sum corresponding to the integral of $f(x) = 2x - 1$ over the interval $[1, 3]$ for the partition $\{1, 3/2, 5/2, 3\}$ with $c_1 = 3/2, c_2 = 2, c_3 = 3$.

Solution. $f(3/2)/2 + f(2) + f(3)/2 = 6.5$.

(b) Sketch the region whose area is given by the integral

$$\int_1^3 |2x - 4|dx.$$

Solution. It is the two right triangles with vertices at $(1, 0), (2, 0), (1, 2)$ and $(2, 0), (3, 0), (3, 2)$.

(15) 8. (a) State the Mean Value Theorem.

Solution. See page 192 of the text.

(b) Prove the following: If $f(x)$ is differentiable on (a, b) and $f'(x) = 0$ on (a, b), then there is a constant C so that $f(x) = C$ on (a, b).

Solution. See page 193 of the text.