(12) 1. Evaluate the following integrals:

$$\int_3^4 \frac{x}{(x^2-4)^{3/2}}dx = \int_5^{12} \frac{1}{2u^{3/2}}du = \frac{1}{\sqrt{5}} - \frac{1}{2\sqrt{3}}.$$

using $u = x^2 - 4, du = 2xdx$.

$$\int \frac{\cos x}{\sin^3 x}dx = \int \frac{1}{u^3}du = -\frac{1}{2u^2} + C = -\frac{1}{2\sin^2 x} + C.$$

using $u = \sin x, du = \cos x dx$.

(13) 2. An object travels in a straight line at velocity $v(t) = t^3 - 3t^2 + 2t$.

(a) Find the displacement over the time period $[0, 3]$.

$$\int_0^3 (t^3 - 3t^2 + 2t)dt = \frac{9}{4}.$$

(b) Find the total distance traveled over the time period $[0, 3]$.

$$\int_0^3 |t^3 - 3t^2 + 2t|dt = \int_0^3 |t(t-1)(t-2)|dt =$$

$$\int_0^1 (t^3 - 3t^2 + 2t)dt - \int_1^2 (t^3 - 3t^2 + 2t)dt + \int_2^3 (t^3 - 3t^2 + 2t)dt = \frac{11}{4}.$$

(12) 3. Sketch the region between the curves with equations $y = x - 2$ and $y = 2x - x^2$, and find its area.

The two curves intersect at the points $(2, 0)$ and $(-1, -3)$. So the area is

$$\int_{-1}^3 (-x^2 + x + 2)dx = \frac{8}{3}$$

(13) 4. Find the volume of the solid whose base is the triangle bounded by the line $x + y = 1$ and the two axes, and whose cross sections perpendicular to the y-axis are semicircles.

$$\int_0^1 \frac{\pi}{2}\left(\frac{1-y}{2}\right)^2 dy = \frac{\pi}{24}.$$

(13) 5. Suppose $x\sqrt{1+2y} + y = x^2$ defines $y = f(x)$ as a function of x near the point $(4, 4)$.

(a) Compute $f'(4)$. Differentiating with respect to x gives

$$\frac{xy'}{\sqrt{1+2y}} + \sqrt{1+2y} + y' = 2x.$$

When $y = x = 4$, this gives $f'(4) = 15/7$.

(b) Find the equation for the tangent line to the graph of f at the point $(4,4)$. Answer: $y = (15x - 32)/7$.

(12) 6. Let $y = x\sqrt{x+1}$. Compute
(a) $y' = (2+3x)/(2\sqrt{x+1})$.
(b) $y'' = (4+3x)/(4(x+1)^{3/2})$.

(13) 7. Use the disk method to compute the volume of the solid obtained by rotating the region bounded by $y = 0, y = 4/(x+1), x = -5$ and $x = -2$ about the x-axis.

$$V = \int_{-5}^{-2} \pi \left(\frac{4}{x+1} \right)^2 dx = 12\pi$$

(12) 8. A rectangle of perimeter 6 is rotated about one of its sides so as to form a cylinder. Among all such rectangles, find the one that maximizes the volume of the cylinder. (Note: the volume of a cylinder whose base is a circle of radius r and has height h is $\pi r^2 h$.)

Suppose the rectangle has sides of lengths x and $3 - x$. If it is rotated about a side of length x, then the resulting cylinder has $r = 3 - x$ and $h = x$, so its volume is $V = \pi(3-x)^2 x$. This is maximized at $x = 1$, so the optimal rectangle has side lengths 1 and 2.

(12) 9. Compute the following derivatives:

$$\frac{d}{dx} \int_0^x \sin^2 t dt = \sin^2 x$$

$$\frac{d}{dx} \int_{x^2}^{x^4} \sqrt{u^2+1} du = 4x^3\sqrt{x^8+1} - 2x\sqrt{x^4+1}.$$

(13) 10. Find the maximum and minimum values of $f(x) = 2\sqrt{x^2+1} - x$ on the interval $[0,2]$.

The derivative is zero when $2x = \sqrt{x^2+1}$, i.e., when $x = 1/\sqrt{3}$. Since $f(0) = 2, f(1/\sqrt{3}) = \sqrt{3}$, and $f(2) = 2\sqrt{5} - 2$, the maximum value is $2\sqrt{5} - 2$ and the minimum value is $\sqrt{3}$.

(15) 11. (a) Find the smallest positive critical point of

$$F(x) = \int_0^x \cos(t^{3/2}) dt.$$

$F'(x) = \cos(x^{3/2})$, so the smallest positive critical point is $a = (\pi/2)^{2/3}$.

(b) Determine whether the critical point you found in part (a) is a local minimum or local maximum.

Since $F''(a) = -(3/2)(\pi/2)^{1/3} < 0$, a is a local max.

(10) 12. Compute the area under the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

$$A = \int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = 2.$$

(12) 13. Compute the following limits:

$$\lim_{x\to 0} \frac{\sin 3x \sin 2x}{x \sin 5x} = \frac{6}{5} \lim_{x\to 0} \frac{\sin 3x}{3x}\frac{\sin 2x}{2x} \Big/ \frac{\sin 5x}{5x} = \frac{6}{5}.$$

$$\lim_{y\to 5} \frac{\frac{y+1}{y-1} - \frac{3}{2}}{y-5} = \lim_{y\to 5} \frac{1}{2-2y} = -\frac{1}{8}.$$

(13) 14. Find the critical points of $y = 2\sin x - \cos^2 x$ on $[0, 2\pi]$, and classify each as a local max, local min, or neither. $y' = 2\cos x(1+\sin x)$, so the critical points are $x = \pi/2$ and $x = 3\pi/2$. $y'' = 2\cos^2 x - 2\sin x - 2\sin^2 x$, which is negative at $x = \pi/2$ and zero at $x = 3\pi/2$. Therefore, $\pi/2$ is a local max, but the second derivative test is inconclusive at $3\pi/2$. Since $1 + \sin x \geq 0$, the sign of y' is the same as the sign of $\cos x$. Therefore, y' changes from negative to positive at $3\pi/2$, so $3\pi/2$ is a local min.

(12) 15. (a) State the Intermediate Value Theorem.

See page 84 of the text.

(b) State the Mean Value Theorem for Integrals.

See page 315 of the text.

(13) 16. Prove the Mean Value Theorem for Integrals.

See page 315 of the text. Alternatively, let

$$F(x) = \int_a^x f(t)dt.$$

Then

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{F(b) - F(a)}{b-a}.$$

By the Mean Value Theorem for derivatives (page 192 of the text), there is a c between a and b so that

$$\frac{F(b) - F(a)}{b-a} = F'(c) = f(c).$$