Mathematics 275B – HW2 – Due Wednesday, February 4, 2009.

A. Problems from the text – to be posted on the web.

B1. (A Borel-Cantelli type lemma) (a) Prove that if W is a nonnegative integer valued random variable, then

$$P(W \geq 1) \geq \frac{(EW)^2}{EW^2}.$$

(b) Suppose A_n are events satisfying (i) $\sum_n P(A_n) = \infty$ and (ii) for some $c > 0$,

$$P(A_n \cap A_m) \leq cP(A_m)P(A_{n-m}) \text{ for } m < n.$$

Show that $P(A_n \ i.o.) > 0$. (Suggestion: Apply part (a) to $W = \sum_{m=1}^n 1_{A_m}$.)

B2. Suppose that $S_n = \sum_{k=1}^n X_k$ is a random walk on the integers satisfying $EX_k = 0$ and $EX_k^2 < \infty$. Show that $P(S_{n^2} = 0 \ i.o.) = 1$. (Suggestion: Use the local limit theorem.)

B3. Suppose S_n is a one dimensional random walk that satisfies

$$S_n \to +\infty \quad a.s.$$

(a) Show that $P(S_n > 0 \text{ for all } n \geq 1) > 0$.

(b) Deduce from this that

$$\sum_n P(S_n \leq 0, S_{n+1} > 0) < \infty,$$

so that the expected number of times it crosses the origin is finite. (Suggestion: Consider the last time that $S_n \leq 0$.)