(20) 1. (a) State some form of Chebyshev's inequality.

(b) Prove the statement in (a).

(c) State the weak law of large numbers.

(d) Prove the weak law of large numbers under a second moment assumption.

(20) 2. Recall that X is Poisson distributed with parameter $\lambda > 0$ if

$$P(X = n) = \frac{e^{-\lambda}\lambda^n}{n!}, \quad n = 0, 1, 2, ...$$

(a) Find the characteristic function for this distribution.

(b) Use characteristic functions to show that if X, Y are independent random variables that are Poisson distributed with parameters λ, μ respectively, then $X + Y$ is Poisson distributed with parameter $\lambda + \mu$.

(c) Suppose that $X(t)$ is Poisson with parameter t. Prove (without using the central limit theorem) that

$$\frac{X(t) - t}{\sqrt{t}} \to N(0, 1)$$

in distribution.

(20) 3. Let X_n be independent random variables, and let $M_n = \max\{X_1, ..., X_n\}$. Prove that $M_n \to \infty$ a.s if and only if

$$\sum_{n=1}^{\infty} P(X_n > t) = \infty \quad \text{for all } t.$$

(20) 4. Let X_n be i.i.d. random variables with a symmetric distribution. Use the Kolmogorov three series theorem to show that $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converges a.s. if and only if $E|X_1| < \infty$.

(20) 5. Let X be an integer valued random variable with characteristic function $\phi(t)$. Prove the inversion formula

$$P(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \phi(t) dt.$$

(20) 6. Suppose X_n are independent positive random variables with finite second moments. Show that if $\sum_{n=1}^{\infty} EX_n = \infty$ and $Var(X_n) \leq CEX_n$ for some constant C, then

$$\frac{S_n}{ES_n} \to 1$$

in probability, where $S_n = X_1 + \cdots + X_n$.

(20) 7. Show that $X_n \to X$ in probability and $Y_n \to Y$ in probability implies $X_n + Y_n \to X + Y$ in probability.

(20) 8. (a) Prove Fatou's lemma for weak convergence: If $\mu_n \to \mu$ weakly on a metric space S and f is nonnegative and continuous, then

$$\int f d\mu \leq \liminf_{n\to\infty} \int f d\mu_n.$$

(b) Show by example that the above inequality may be strict.

(20) 9. State and prove the Kolmogorov zero-one law.

(20) 10. (a) State the central limit for sums S_n of i.i.d. random variables.

(b) Prove that under the assumptions of this theorem (and assuming that the summands have mean zero and variance 1),

$$\lim_{n\to\infty} \frac{E|S_n|}{\sqrt{n}}$$

exists, and find its limit.