T. Liggett Mathematics 171 – Midterm 1 Solutions April 25, 2011

The hitting time $\min\{n \geq 1 : X_n = x\}$ will be denoted by T_x.

(24) 1. For each statement below, say whether it is true or false. No explanation is necessary. (Scoring: +3 if correct, −1 if incorrect, 0 if no answer.)

(a) For a finite state Markov chain, some state is recurrent. True, since if all states were transient, each of the finitely many states would be visited only finitely many times, and this would account for only finitely many time steps. However, there are infinitely many time steps.

(b) For an infinite state irreducible Markov chain, every state is transient. False. Example, simple symmetric random walk on the integers. In this case, all states are recurrent.

(c) Every irreducible Markov chain has a stationary distribution. False. Simple random walk on the integers has no stationary distribution.

(d) If the Markov chain with transition matrix P is irreducible, then so is the Markov chain with transition matrix P^2. False. Again, consider simple random walk on the integers. It is irreducible, but the chain with matrix P^2 cannot go from an odd state to an even state. Also, see problem 9.12.

(e) If π is stationary and $p(x, y) > 0$ for all x, y, then π satisfies the detailed balance condition. False. Consider for example a doubly stochastic, finite state, irreducible Markov chain with all entries strictly positive. Then the stationary distribution is the uniform distribution. It satisfies detailed balanced if and only if P is symmetric, which it need not be.

(f) If x is transient and $X_0 = x$, then $N =$ (the number of $n \geq 0$ such that $X_n = x$) has a geometric distribution. True (It was pointed out to me that there is some ambiguity about this – whether the degenerate distribution =1 is regarded as being a geometric. Therefore all students will receive 3 points for this.) This is true provided one includes degenerate geometrics as being geometric.

(g) If $S =$ the set of integers and $p(x, x + 1) = p(x, x - 1) = \frac{1}{2}$ for each x then the chain is recurrent. True. We proved this in class in at least two different ways.

(h) If $S =$ the set of integers and $p(x, x + 1) = p(x, x - 1) = \frac{1}{2}$ for each x then the chain is aperiodic. False. It has period 2.

(20) 2. (a) Define "x communicates with y" ($x \to y$).

Solution: $p^{(n)}(x, y) > 0$ for some n

(b) Assume that $x \to y$ and $y \to x$. Prove that if x is transient, then so is y.

Solution: By Chapman-Kolmogorov,

$$p^{(k+l+m)}(x,x) \geq p^{(k)}(x,y)p^{(l)}(y,y)p^{(m)}(y,x).$$

Summing on l gives

$$G(x,x) \geq p^{(k)}(x,y)G(y,y)p^{(m)}(y,x),$$

where $G(x,x)$ is the expected number of visits to x starting at x. Choose k,m so that $p^{(k)}(x,y) > 0, p^{(m)}(y,x) > 0$. If x is transient, then $G(x,x) < \infty$. It follows that $G(y,y) < \infty$, and then that y is transient.

(26) 3. Consider the Markov chain with state space $S = \{1,2,3,4,5,6\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

(a) What is the decomposition of S into transient states and closed irreducible recurrent classes?

Solution: $S = T \cup R_1 \cup R_2$, where $T = \{5,6\}$, $R_1 = \{1,3\}$ and $R_2 = \{2,4\}$.

(b) Find $\lim_{n\to\infty} p^{(n)}(1,1)$.

Solution: The equations for the stationary distribution of the chain restricted to R_1 are $\pi(1)\frac{1}{2} + \pi(3)\frac{1}{4} = \pi(1)$ and $\pi(1)\frac{1}{2} + \pi(3)\frac{3}{4} = \pi(3)$. The solution is $\pi(1) = \frac{1}{3}, \pi(3) = \frac{2}{3}$. Therefore, $\lim_{n\to\infty} p^{(n)}(1,1) = \frac{1}{3}$.

(c) Find $\lim_{n\to\infty} p^{(n)}(5,1)$.

Solution: Let $h(x)$ be the probability of eventual absorption into R_1 starting from x. Then $h(5) = \frac{3}{8} + \frac{1}{4}h(5) + \frac{1}{8}h(6), h(6) = \frac{1}{3} + \frac{1}{6}h(5) + \frac{1}{6}h(6)$. The solution to these equations is $h(5) = \frac{17}{29}, h(6) = \frac{15}{29}$. Therefore $\lim_{n\to\infty} p^{(n)}(5,1)\frac{17}{29}\frac{1}{3} = \frac{17}{87}$.

(d) Find E_5N, where R is the set of recurrent states and

$$N = \min\{n \geq 1 : X_n \in R\}.$$

Solution: Now let $h(x) = E_xN$. Then $h(5) = 1 + \frac{1}{4}h(5) + \frac{1}{8}h(6), h(6) = 1 + \frac{1}{6}h(5) + \frac{1}{6}h(6)$. The solution to these equations is $h(5) = \frac{46}{29}, h(6) = \frac{44}{29}$. Therefore, $E_5N = \frac{46}{29}$.

(30) 4. The renewal chain X_n has state space $\{0, 1, 2, \dots\}$ and transition probabilities $p(n, n-1) = 1$ for $n \geq 1$ and $p(0, n) = p_n$, where $p_n \geq 0$ satisfies $\sum_{n=0}^{\infty} p_n = 1$.

(a) What are necessary and sufficient conditions for the chain to be irreducible? Explain.

Solution: Infinitely many of the p_n's need to be positive. If only finitely many are positive, and p_m is the last positive one, then 0 does not communicate with $m+1$. If infinitely many are positive, to show that m communicates with n, let $l > n$ be such that $p_l > 0$. Then $p^{(l+m-n+1)}(m, n) \geq p_l > 0$.

Assume now that the chain is irreducible.

(b) Under what conditions is it recurrent? Explain.

Solution: It is always recurrent, since $P_0(T_0 = n) = p_{n-1}$ for $n \geq 1$, so $P_0(T_0 < \infty) = \sum_{n=1}^{\infty} p_{n-1} = 1$.

(c) Compute E_0T_0.

Solution: $E_0T_0 = \sum_{n=1}^{\infty} np_{n-1} = \sum_{n=0}^{\infty} np_n + 1$.

(d) Under what conditions is the chain positive recurrent? Explain.

Solution: It is positive recurrent if and only if $\sum_n np_n < \infty$, since this is equivalent to $E_0T_0 < \infty$.

(e) In the positive recurrent case, compute the stationary distribution $\pi(k), k \geq 0$.

Solution: The stationary distribution satisfies $\pi(n) = \pi(0)p_n + \pi(n+1)$. Rewrite this as

$$\pi(n) - \pi(n+1) = \pi(0)p_n,$$

and sum from k to ∞. The result is

$$\pi(k) = \pi(0) \sum_{n=k}^{\infty} p_k.$$

By part (c), it then follows that

$$\pi(k) = \sum_{n=k}^{\infty} p_k \Bigg/ \left(\sum_{n=0}^{\infty} n p_n + 1 \right).$$