Mathematics 171 – HW5 – Due Thursday, May 5, 2011.

Problems 5.1,5.2,5.3,5.4,5.5 on page 121, plus the following:

G. Consider a finite state Markov chain satisfying $p(x,y) > 0$ for all $x \neq y$. Show that its stationary distribution satisfies the detailed balance condition if and only if

$$p(x,y)p(y,z)p(z,x) = p(x,z)p(z,y)p(y,x)$$

for all x, y, z.

H. Consider the branching chain discussed in class on April 20, and in the text as Example 7.2. Let μ and σ^2 be the mean and variance of the offspring distribution.

(a) Show that $E_k X_1^2 = k\sigma^2 + k^2\mu^2$.

(b) Show that $E_k X_{n+1}^2 = k\mu^n\sigma^2 + \mu^2 E_k X_n^2$.

(c) Use part (b) repeatedly to compute $E_k X_n^2$.

(d) Compute the variance of X_n if $X_0 = k$.

I. Recall the Schwarz inequality: $(EXY)^2 \leq EX^2 EY^2$. Show that if $X \geq 0$, then

$$E(X \mid X > 0) \leq \frac{EX^2}{EX}.$$

J. Let X_n be the branching chain with offspring distribution $p_k = qp^k$ for $k \geq 0$, where $p + q = 1$ and $p > q$. Let $M_n = X_n/\mu^n$ be the associated martingale.

(a) Use problems H and I to show that

$$E_1(M_n \mid M_n > 0) \leq \frac{2p}{p-q}.$$

(b) Show that

$$\lim_{n\to\infty} E_1(M_n \mid M_n > 0) = \frac{p}{p-q}.$$