T. Liggett Mathematics 171 – Midterm 2 May 16, 2011

(15) 1. Let X_n be a Markov chain taking values in $[0,1]$ with the following transition mechanism: Conditional on the values of $X_0 \dots X_n$

$$X_{n+1} = \begin{cases} a + (b-a)X_n & \text{with probability } X_n; \\ c + (d-c)X_n & \text{with probability } 1 - X_n, \end{cases}$$

where $a, b, c, d \in [0,1]$. Under what conditions is X_n a martingale? Explain.

Solution:

$$\begin{aligned} E(X_{n+1} \mid X_0, X_1, \dots, X_n) &= (a + (b-a)X_n)X_n + (c + (d-c)X_n)(1 - X_n) \\ &= c + (a + d - 2c)X_n + (b - a - d + c)X_n^2, \end{aligned}$$

so $c = 0, a + d = 1, b = 1$. (Note: This the type of martingale that appears in the Polya urn application.)

(20) 2. Suppose $N(t)$ is a rate λ Poisson process.

(a) Compute $EN(s)N(t+s)$.

Solution: $EN(s)N(t+s) = EN^2(s) + EN(s)[N(t+s) - N(s)] = \lambda s + (\lambda s)^2 + \lambda^2 st$.

(b) Find $P(N(2) = 3 \mid N(6) = 6)$.

Solution:

$$P(N(2) = 3 \mid N(6) = 6) = P(N(2) = 3, N(6) - N(2) = 3)/P(N(6) = 6) = \binom{6}{3}(\frac{1}{3})^3(\frac{2}{3})^3.$$

(30) 3. Consider the Markov chain on $\{0, 1, \dots\}$ with transition probabilities

$$p(k, k+1) = \frac{1}{k+1}, \quad p(k, 0) = \frac{k}{k+1}$$

for $k \geq 0$.

(a) Find the stationary distribution for the chain.

Solution: The equations for π are $\pi(k) = \pi(k-1)/k$ for $k \geq 1$. Solving gives $\pi(k) = \pi(0)/(k!)$. So, the stationary distribution is the Poisson distribution with parameter 1.

(b) What is the value of $E_0 T_0$?

Solution: $E_0 T_0 = 1/\pi(0) = e$.

(c) Let $u(k) = E_k T_0$. Find an equation that relates $u(k)$ to $u(k+1)$.

Solution: $u(k) = 1 + \frac{u(k+1)}{k+1}$.

(d) Compute $u(2)$.

Solution: Solving recursively gives $u(0) = e, u(1) = e - 1, u(2) = 2e - 4$. (Note: While I did not ask you to do it, you can easily check by induction that

$$u(n) = n! \sum_{k=n}^{\infty} \frac{1}{k!} = \frac{P(X \geq n)}{P(X = n)} = \frac{1}{P(X = n \mid X \geq n)},$$

where X is Poisson with parameter 1.)

(20) 4. Consider the Markov chain X_n on $\{0, 1, \dots\}$ with transition probabilities

$$p(k, k+1) = p_k, \quad p(k, 0) = 1 - p_k$$

for $k \geq 0$.

(a) For what choice(s) of the sequence p_k is X_n a martingale?

Solution: $E(X_{n+1} \mid X_0 = k_0, \dots, X_n = k) = (k+1)p_k$, so we must take $p_k = \frac{k}{k+1}$.

(b) For the choice(s) you found in (a), use the stopping time theorem to compute $P_k(T_N < T_0)$ for $0 < k < N$.

Solution: Let $T = \min(T_0, T_N)$. Then $k = E_k X_T = NP(X_T = N) + 0P(X_T = 0)$, so $P_k(T_N < T_0) = \frac{k}{N}$.

(15) 5. Orders for three products A,B and C come in to a warehouse according to independent Poisson processes of rates per day of 1,2, and 3 respectively.

(a) What is the expected total number of orders received in a given day.

Solution: 1+2+3=6.

(b) What is the probability that the first order received after noon that day is for product B?

Solution: $\frac{2}{1+2+3} = \frac{1}{3}$.

(c) What is the expected time between noon that day and the time of the first order for any of the products after noon.

Solution: The elapsed time is Exponential with parameter 6; hence the expected time is $\frac{1}{6}$.