T. Liggett Mathematics 170A – Midterm 2 Solutions March 7, 2012

1. (15) The continuous random variable X has PDF

$$f(x) = \begin{cases} cx(1-x) & \text{if } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c.

Solution: $\int_0^1 x(1-x)dx = \frac{1}{6}$, so $c = 6$.

(b) Find EX.

Solution: $EX = \int_0^1 6x^2(1-x)dx = \frac{1}{2}$, or use symmetry.

(c) Find Var(X).

Solution:$EX^2 = \int_0^1 6x^3(1-x)dx = \frac{3}{10}$, so $\mathrm{Var}(X) = \frac{1}{20}$.

2. (15) State whether each of the following is true or false. No explanation is required. (Scoring: +3 for a correct answer, −1 for an incorrect answer, 0 for no answer.)

(a) If X and Y are independent random variables with finite second moments, then they are uncorrelated.

Solution: True.

(b) If X and Y are uncorrelated random variables, then they are independent.

Solution: False.

(c) If X is a nonnegative random variable, then $EX < \infty$.

Solution: False.

(d) There exists a random variable with $EX = 5, EX^2 = 2$.

Solution: False.

(e) There exists a random variable with $EX = 2, EX^2 = 5$.

Solution: True.

3. (15) Find the PMF of $X + Y$ where X and Y are independent geometric random variables with parameters p and q respectively, when

(a) $p = q$.

Solution:

$$\begin{aligned} P(X+Y=n) &= \sum_{k=1}^{n-1} P(X=k)P(Y=n-k) \\ &= \sum_{k=1}^{n-1} p^2(1-p)^{n-2} = (n-1)p^2(1-p)^{n-2}. \end{aligned}$$

(b) $p \neq q$.

Solution:

$$\begin{aligned} P(X+Y=n) &= \sum_{k=1}^{n-1} P(X=k)P(Y=n-k) = \sum_{k=1}^{n-1} pq(1-p)^{k-1}(1-q)^{n-k-1} \\ &= \frac{pq}{p-q}[(1-q)^{n-1}-(1-p)^{n-1}]. \end{aligned}$$

4. (15) 72 people each toss a fair die twice. Let X be the number of people who get snake eyes (i.e., a 1 on each die).

(a) What is $P(X = 3)$?

Solution: X is $\mathrm{B}(72, \frac{1}{36})$, so

$$P(X=3) = \binom{72}{3}\left(\frac{1}{36}\right)^3\left(\frac{35}{36}\right)^{69}.$$

(b) Use the Poisson approximation to approximate $P(X = 3)$.

Solution: X is approximately Poisson (2), so $P(X = 3) \sim e^{-2}2^3/3!$.

5. (20) Let $X_1, X_2, \ldots$ be independent random variables with the $B(5, \frac{1}{2})$ distribution, and T be a random variable independent of the X_i's with the geometric distribution with parameter p. Let $S_n = X_1 + X_2 + \cdots + X_n$, so $S_T = X_1 + X_2 + \cdots + X_T$.

(a) What is the distribution of S_n?

Solution: $B(5n, \frac{1}{2})$.

(b) What is EX_i?

Solution: $\frac{5}{2}$.

(c) Find $E(S_T \mid T = n)$.

Solution: $\frac{5}{2}n$.

(d) Find ES_T.

Solution:

$$ES_T = \sum_{n=1}^{\infty} E(S_T \mid T=n)P(T=n) = \sum_{n=1}^{\infty} \frac{5}{2}nP(T=n) = \frac{5}{2}ET = \frac{5}{2p}.$$

6. (20) Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli random variables with parameter p (i.e., they are independent and satisfy $P(X_i = 1) = p, P(X_i = 0) = 1 - p$). Define $Y_i = X_i X_{i+1}$ and $S_n = Y_1 + Y_2 + \cdots + Y_n$.

(a) Find EY_i.

Solution: $EY_i = EX_i EX_{i+1} = p^2$.

(b) Find EY_iY_j for all pairs $i, j \geq 1$.

Solution:

$$EY_iY_j = \begin{cases} p^2 & \text{if } i = j; \\ p^3 & \text{if } |i - j| = 1; \\ p^4 & \text{otherwise.} \end{cases}$$

(c) Find ES_n.

Solution: $ES_n = np^2$.

(d) Find $\mathrm{Var}(S_n)$.

Solution: Since Y_i and Y_j are independent for $|i - j| > 1$,

$$\begin{aligned} \mathrm{Var}(S_n) &= \sum_{1 \leq i,j \leq n} \mathrm{Cov}(Y_i, Y_j) = n\mathrm{Var}(Y_1) + 2(n-1)\mathrm{Cov}(Y_1, Y_2) \\ &= np^2(1 - p^2) + 2(n-1)p^3(1 - p) = p^2(1 - p)[n(1 + 3p) - 2p]. \end{aligned}$$