T. Liggett Mathematics 170A – Midterm 1 February 8, 2012

1	2	3	4	5	6	Total

Last name: First name:

(15) 1. A fair die is tossed twice. Let S be the sum of the two outcomes, $A = \{S \text{ is even}\}$ and $B = \{S \le 6\}$.

(a) Find $P(A)$ and $P(B)$.

Solution: $P(A) = \frac{2(1+3+5)}{36} = \frac{1}{2}, P(B) = \frac{1+2+3+4+5}{36} = \frac{5}{12}$.

(b) Find $P(A \mid B)$.

Solution: $P(A \cap B) = \frac{1+3+5}{36} = \frac{1}{4}$, so $P(A \mid B) = \frac{3}{5}$.

(c) Are A and B independent? Explain.

Solution: No, since $P(A \mid B) \neq P(A)$.

(15) 2. A box contains 10 red balls and 5 black balls. A ball is selected at random from the box. If the ball is red, it is simply returned to the box. If it is black, it and three additional black balls are added to the box. Then a second ball is drawn from the box.

(a) Find $P(\text{the second ball is red})$.

Solution: Let $A = \{\text{first is red}\}$ and $B = \{\text{second is red}\}$. Then

$$P(B) = P(B \mid A)P(A) + P(B \mid A^c)P(A^c) = \frac{2}{3}\frac{2}{3} + \frac{5}{9}\frac{1}{3} = \frac{17}{27}.$$

(b) Find $P(\text{the first ball was red} \mid \text{the second ball is red})$.

Solution: $P(A \mid B) = P(B \mid A)P(A)/P(B) = \frac{12}{17}$.

(15) 3. Suppose $A_1, A_2, \dots, A_n$ are independent events with $P(A_i) = p_i$.

(a) Find $P(A_1 \cup A_2)$.

Solution: $P(A_1 \cup A_2) = p_1 + p_2 - p_1 p_2 = 1 - (1 - p_1)(1 - p_2)$.

(b) Find $P(A_1 \cup A_2 \cup \cdots \cup A_n)$.

Solution:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - P(A_1^c \cap A_2^c \cap \cdots \cap A_n^c) = 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n).$$

(20) 4. A hand of 13 cards is drawn from a standard 52 card deck.

(a) Find the probability that it consists of 4 spades, 4 hearts, 3 diamonds and 2 clubs.

Solution:

$$\frac{\binom{13}{4}^2\binom{13}{3}\binom{13}{2}}{\binom{52}{13}}.$$

(b) Find the probability that it consists of 4 cards of each of two suits, 3 of another, and 2 of the other.

Solution:

$$\binom{4}{2}2\frac{\binom{13}{4}^2\binom{13}{3}\binom{13}{2}}{\binom{52}{13}}.$$

(15) 5. A population consists of three types of individuals: 25 of type 1, 55 of type 2 and 70 of type 3. A sample of size 20 is taken without replacement. Let A, B, C be the number of types 1,2,3 respectively in the sample.

(a) Find $P(A = 10)$.

Solution:

$$\frac{\binom{25}{10}\binom{125}{10}}{\binom{150}{20}}.$$

(b) Find $P(A = 10 \text{ and } B = 5)$.

Solution:

$$\frac{\binom{25}{10}\binom{55}{5}\binom{70}{5}}{\binom{150}{20}}.$$

(20) 6. A fair coin is tossed repeatedly. Let N the the number of the toss on which the first head appears.

(a) Find $P(N = 6)$.

Solution: $P(N = 6) = \frac{1}{2^6}$.

(b) Let A be the event that N is a multiple of 3. Find $P(A)$.

Solution:

$$P(A) = \sum_{n=1}^{\infty} \frac{1}{2^{3n}} = \sum_{n=1}^{\infty} \frac{1}{8^n} = \frac{1}{8}\frac{1}{1-\frac{1}{8}} = \frac{1}{7}.$$