(12) 1. Suppose X is normally distributed with mean 2 and variance 9.

(a) Find $P(X \geq 7)$.

Solution:

$$P(X \geq 7) = P\left(\frac{X-2}{3} \geq \frac{5}{3}\right) = 1 - \Phi(5/3) = 1 - .9525 = .0475.$$

(b) Find a number c so that $P(X \leq c) = .10$.

$$.10 = P(X \leq c) = P\left(\frac{X-2}{3} \leq \frac{c-2}{3}\right) = \Phi((c-2)/3).$$

So, choose c so that $(c-2)/3 = -.8159$, i.e., $c = -.4477$.

Solution:

(20) 2. A point (X, Y) is chosen uniformly from the square $[-1, 1]^2$.

(a) Compute $P(Y > X^2)$.

Solution:

$$P(Y > X^2) = \frac{1}{4}\int_{-1}^{1}(1 - x^2)dx = \frac{1}{3}.$$

(b) Find the distribution function (CDF) of $Z = |X| + |Y|$.

Solution:

$$P(Z \leq z) = \begin{cases} 0 & \text{if } z \leq 0; \\ \frac{1}{2}z^2 & \text{if } 0 \leq z \leq 1; \\ -1 + 2z - \frac{1}{2}z^2 & \text{if } 1 \leq z \leq 2; \\ 1 & \text{if } z \geq 2. \end{cases}$$

(c) Find the density function (PDF) of Z.

Solution:

$$f(z) = \begin{cases} z & \text{if } 0 \leq z \leq 1; \\ 2 - z & \text{if } 1 \leq z \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

(15) 3. Find the PMF of $X + Y$ where X and Y are independent geometric random variables with parameters p and q respectively, when

(a) $p = q$.

Solution:

$$\begin{aligned} P(X+Y=n) &= \sum_{k=1}^{n-1} P(X=k)P(Y=n-k) \\ &= \sum_{k=1}^{n-1} p^2(1-p)^{n-2} = (n-1)p^2(1-p)^{n-2}. \end{aligned}$$

(b) $p \neq q$.

Solution:

$$\begin{aligned} P(X+Y=n) &= \sum_{k=1}^{n-1} P(X=k)P(Y=n-k) = \sum_{k=1}^{n-1} pq(1-p)^{k-1}(1-q)^{n-k-1} \\ &= \frac{pq}{p-q}[(1-q)^{n-1} - (1-p)^{n-1}]. \end{aligned}$$

(20) 4. Suppose X is a continuous random variable with PDF

$$f(x) = \begin{cases} 6x(1-x) & \text{if } 0 \leq x \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then X divides the unit interval $[0,1]$ into two parts. Let $Y \geq 1$ be the ratio of the longer part to the shorter part.

(a) Find $P(Y \leq 2)$.

Solution:

$$P(Y \leq 2) = P\left(\frac{1}{3} \leq X \leq \frac{2}{3}\right) = \frac{13}{27}.$$

(b) Find the PDF of Y.

Solution: For $y \geq 1$,

$$P(Y \leq y) = P\left(\frac{1}{y+1} \leq X \leq \frac{y}{y+1}\right) = \int_{\frac{1}{y+1}}^{\frac{y}{y+1}} f(x)dx = \frac{(y-1)(y^2+4y+1)}{(y+1)^3}.$$

(c) Find EY.

Solution:

$$EY = \int_0^\infty P(Y \geq y)dy = 1 + \int_1^\infty \frac{2(3y+1)}{(y+1)^3}dy = \frac{7}{2}.$$

(18) 5. Short answer questions. No explanation necessary.

(a) Suppose X has the $N(-1, 9)$ distribution. What is the distribution of $2X + 5$?

Solution: $N(3, 36)$.

(b) Suppose X satisfies $EX = -5$ and $EX^2 = 25$. What can you say about X?

Solution: $P(X = -5) = 1$.

(c) What value of a minimizes $E(X - a)^2$?

Solution: $a = EX$.

(d) If (X_1, X_2, X_3, X_4) has the multinomial distribution with parameters n and $p_1 = \frac{1}{6}, p_2 = \frac{1}{3}, p_3 = \frac{1}{4}, p_4 = \frac{1}{4}$, what is the distribution of $X_1 + X_3$?

Solution: $B(n, \frac{5}{12})$.

(e) Suppose the events A and B are both disjoint and independent. What can you conclude about A and B?

Solution: $P(A) = 0$ or $P(B) = 0$.

(f) If (X, Y) is uniformly distributed on a two-dimensional connected region G, and X and Y are independent, what can you conclude about G?

Solution: G is a rectangle.

(15) 6. Suppose that X has the $N(0, 1)$ distribution.

(a) Find the density function of $|X|$.

Solution: For $x \geq 0$, $P(X \leq x) = \Phi(x) - \Phi(-x)$, so the density is

$$\sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2}, x \geq 0.$$

(b) Compute Ee^X.

Solution:

$$Ee^X = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \sqrt{e} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-1)^2} dx = \sqrt{e}.$$

(20) 7. Consider a collection of n married couples. A sample of size $m \leq 2n$ is taken from the $2n$ people without replacement. Let X be the number of married couples in the sample.

(a) Find the PMF of X.

Solution:

$$P(X = k) = \frac{\binom{n}{k}\binom{n-k}{m-2k} 2^{m-2k}}{\binom{2n}{m}}.$$

(b) Use indicator random variables to compute EX.

Solution: For $1 \leq i \leq n$, let

$$X = \begin{cases} 1 & \text{if the } i\text{th couple is in the sample;} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P(X_i = 1) = \frac{\binom{2n-2}{m-2}}{\binom{2n}{m}} = \frac{m(m-1)}{2n(2n-1)},$$

so

$$EX = \frac{m(m-1)}{2(2n-1)}.$$

(20) 8. Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli random variables with parameter p (i.e., they are independent and satisfy $P(X_i = 1) = p, P(X_i = 0) = 1 - p$). Define $Y_i = X_i X_{i+1} X_{i+2}$ and $S_n = Y_1 + Y_2 + \cdots + Y_n$.

(a) Find EY_i.

Solution: $EY_i = p^3$.

(b) Find EY_iY_j for **all** pairs $i, j \geq 1$.

Solution:

$$EY_iY_j = \begin{cases} p^3 & \text{if } i = j; \\ p^4 & \text{if } |i - j| = 1; \\ p^5 & \text{if } |i - j| = 2; \\ p^6 & \text{otherwise.} \end{cases}$$

(c) Find ES_n.

Solution: $ES_n = np^3$.

(d) Find $\text{Var}(S_n)$.

Solution: $\text{Var}(S_n) = np^3(1-p^3) + 2(n-1)p^4(1-p^2) + 2(n-2)p^5(1-p)$.

(20) 9. The joint probability density function of (X, Y) is given by

$$f(x,y) = \begin{cases} c(x+y)^2 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c.

Solution: $c = \frac{6}{7}$.

(b) Are X and Y independent? Explain.

Solution: No, since $f(x, y)$ cannot be written as the product of a function of x and a function of y.

(c) Let $U = X + Y$ and $V = X - Y$. Find the joint density function $g(u, v)$ of (U, V). Be sure to specify the values of (u, v) for which $g(u, v) > 0$.

Solution: $g(u, v) = \frac{3}{7}u^2$ for (u, v) in the square with vertices

$$(0,0), (2,0), (1,1), (1,-1).$$

(15) 10. Suppose $X_1, X_2, \ldots, X_n$ are independent continuous random variables, each having distribution function $F(x)$ and density function $f(x)$. Let $Y = \max(X_1, \ldots, X_n)$.

(a) Find the distribution function of Y.

Solution: $P(Y \le y) = P(X_1 \le y, \ldots, X_n \le y) = [F(y)]^n$.

(b) Find the density function of Y.

Solution: The density function $g(y)$ of Y is

$$g(y) = \frac{d}{dy}[F(y)]^n = n[F(y)]^{n-1} f(y).$$

(15) 11. Box I contains 2 white balls and 2 black balls, box II contains 2 white balls and 1 black ball, and box III contains 1 white ball and 3 black balls. One of the three boxes is chosen at random, and one ball is drawn from it.

(a) What is the probability that the ball drawn is white?

Solution: Let $A = \{\text{the ball drawn is white}\}$. Then $P(A) =$

$$P(A \mid I)P(I) + P(A \mid II)P(II) + P(A \mid III)P(III) = \frac{1}{3}\left[\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right] = \frac{17}{36}.$$

(b) Given that the ball drawn is white, what is the probability that the first box was chosen?

Solution:

$$P(I \mid A) = \frac{P(A \mid I)P(I)}{P(A)} = \frac{6}{17}.$$

(10) 12. Two fair dice are thrown, and let S be sum of the outcomes. Consider the events

$$A = \{S = 2, 3, \text{ or } 4\}, \quad B = \{S = 3, 5, \text{ or } 7\}, \quad C = \{S = 4, 7, 8, \text{ or } 9\}.$$

(a) Are A and B independent? Explain.

Solution: Yes, since $P(A) = \frac{1}{6}, P(B) = \frac{1}{3}, P(A \cap B) = \frac{1}{18} = P(A)P(B)$.

(b) Are A, B, C independent? Explain.

Solution: No, since $A \cap B \cap C = \emptyset$, so $0 = P(A \cap B \cap C) \neq P(A)P(B)P(C)$.