(25) 1. (a) State Fatou's Lemma.

See Royden, page 86.

(b) State the Bounded Convergence Theorem.

See Royden, page 84.

(c) Use Fatou's Lemma to prove the Bounded Convergence Theorem.

Suppose $|f_n| \le M$ for each n. Then $M + f_n$ and $M - f_n$ are nonnegative functions. By Fatou,

$$Mm(E) - \int_E f = \int_E (M - f) \le \liminf_{n\to\infty} \int_E (M - f_n) = Mm(E) - \limsup_{n\to\infty} \int_E f_n$$

and

$$Mm(E) + \int_E f = \int_E (M + f) \le \liminf_{n\to\infty} \int_E (M + f_n) = Mm(E) + \liminf_{n\to\infty} \int_E f_n.$$

Since $m(E) < \infty$,

$$\limsup_{n\to\infty} \int_E f_n \le \int_E f \le \liminf_{n\to\infty} \int_E f_n,$$

so

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

(20) 2. Evaluate

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{ax} dx$$

for each real a. Justify each of your steps.

The integrand converges to $e^{(a-1)x} 1_{[0,\infty)}(x)$ for each x. Since $1 - t \le e^{-t}$ for all t, the integrand is dominated by $e^{(a-1)x} 1_{[0,\infty)}(x)$, which is integrable for $a < 1$. By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{ax} dx = \int_0^\infty e^{(a-1)x} dx = \frac{1}{1-a}$$

for $a < 1$. Since

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{ax} dx$$

is increasing in a, it follows that

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{ax} dx = \infty$$

for $a \geq 1$.

(20) 3. Suppose that f is increasing on $[0,1]$. Using the fact that f is a.e. differentiable, show that

$$\int_0^1 f'(x) \leq f(1) - f(0).$$

Let $f(x) = f(1)$ for $x \geq 1$, and then

$$f_n(x) = n\left[f\left(x + \frac{1}{n}\right) - f(x)\right].$$

Note that $f_n \geq 0$, $f_n \to f'$ a.e., and

$$\int_0^1 f_n(x)dx = n\int_1^{1+\frac{1}{n}} f(x)dx - n\int_0^{\frac{1}{n}} f(x)dx \leq f(1) - f(0).$$

Therefore, the statement follows from Fatou.

(20) 4. Consider the system of equations

$$x + xyz = u, \quad y + xy = v, \quad z + 2x + 3z^2 = w.$$

Can these be solved locally near $(0,0,0)$? Explain.

Let $f(x,y,z) = (x + xyz, y + xy = v, z + 2x + 3z^2)$. Then

$$f'(0,0,0) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is invertible. So, the inverse function theorem guarantees local solvability.

(15) 5. Suppose that f is integrable on $[0,1]$, and $\int_A f(x)dx = 0$ for every measurable subset A of $[0,1]$. Show that $f = 0$ a.e. on $[0,1]$.

Let $A_n = \{x : f(x) \geq \frac{1}{n}\}$. Then

$$m(A_n) \leq n\int_{A_n} f(x)dx = 0.$$

By countable subadditivity,

$$m(\{x : f(x) > 0\}) \le \sum_n m(A_n) = 0,$$

so $f \le 0$ a.e. Similarly, $f \ge 0$ a.e.

(25) 6. Consider the integration by parts formula

$$\int_a^b F(x)G'(x)dx = -\int_a^b F'(x)G(x)dx + F(b)G(b) - F(a)G(a). \qquad (1)$$

(a) Give an example of continuous functions of bounded variation F and G for which (1) is false.

Let $F(x) = G(x) =$ the Cantor function, with $a = 0, b = 1$.

(b) Show that the product of absolutely continuous functions is absolutely continuous.

Suppose that F and G are absolutely continuous. Then they are continuous on $[a, b]$ and hence bounded, say by M. Given $\epsilon > 0$, let $\delta > 0$ be the smaller of the values that appear in the definition of absolute continuity for F and G for that ϵ. If $[x_i, y_i], 1 \le i \le n$, are disjoint with $\sum_i (y_i - x_i) < \delta$, then

$$\sum_i |F(y_i) - F(x_i)| < \epsilon \text{ and } \sum_i |G(y_i) - G(x_i)| < \epsilon.$$

It follows that

$$\sum_i |F(y_i)G(y_i) - F(x_i)G(x_i)| \le$$

$$\sum_i |F(y_i)|G(y_i) - G(x_i)| + \sum_i |F(y_i) - F(x_i)|G(x_i)| \le 2M\epsilon.$$

Now replace ϵ by $\epsilon/(2M)$.

(c) Prove (1) for absolutely continuous F and G.

By part (b), FG is absolutely continuous. Therefore,

$$\int_a^b \frac{d}{dx}[F(x)G(x)]dx = F(b)G(b) - F(a)G(a).$$

Now apply the product rule.

(10) 7. Compute the total variation $T_0^{2\pi} f$ and positive variation $P_0^{2\pi} f$ of the function $f(x) = \sin x$ on the interval $[0, 2\pi]$.

f is increasing on $[0, \frac{\pi}{2}]$, decreasing on $[\frac{\pi}{2}, \frac{3\pi}{2}]$, and increasing on $[\frac{3\pi}{2}, 2\pi]$. The max and min values on $[0, 2\pi]$ are ± 1. Therefore $T_0^{2\pi} f = 4$ and $P_0^{2\pi} f = 2$.

(20) 8. Suppose $f : X \to X$, where (X, d) is a metric space, satisfies

$$d(f(x), f(y)) \leq c \, d(x, y) \quad \text{for all} \quad x, y \in X.$$

(a) Show that if $c = \frac{1}{2}$, the sequence defined by $x_1 \in X$ and $x_{n+1} = f(x_n)$ for $n \geq 1$ is Cauchy.

See Rudin, page 220.

(b) Give an example to show that the statement in (a) is false if $c = 1$.

$X = R^1, f(x) = x + 1$.

(25) 9. Let l_1 be the space of all sequences $f = (f(1), f(2), ...)$ so that

$$||f|| = \sum_{k=1}^{\infty} |f(k)| < \infty.$$

(a) Suppose that $f_n \in l_1$ for each n, and that there is a constant C so that $||f_n|| \leq C$ for each n. Show that there is a subsequence f_{n_i} so that $f(k) = \lim_{i \to \infty} f_{n_i}(k)$ exists for each k, and that the resulting f is in l_1.

For each k, the set $\{f_n(k), n \geq 1\}$ is bounded by C. Therefore, the subsequence can be constructed by a diagonal argument. Then

$$\sum_{k=1}^{m} |f(k)| = \lim_{i} \sum_{k=1}^{m} |f_{n_i}(k)| \leq C$$

for each m, so $||f|| \leq C$.

(b) Prove or disprove that the subsequence in (a) can always be chosen so that $f_{n_i} \to f$ in l_1.

This is false. Example: $f_n(n) = 1, f_n(k) = 0$ for all $k \neq n$. Then $||f_n|| = 1$ for each n, but for any subsequence, the coordinate-wise limit $f = 0$.

(c) Prove that l_1 is complete.

Suppose f_n is Cauchy in l_1. Then for each k, $f_n(k)$ is a Cauchy sequence in R^1, so $f(k) = \lim_{n\to\infty} f_n(k)$ exists. Then for each l,

$$\sum_{k=1}^{l} |f(k) - f_n(k)| = \lim_{m\to\infty} \sum_{k=1}^{l} |f_m(k) - f_n(k)|,$$

so

$$||f - f_n|| \leq \limsup_{m\to\infty} ||f_m - f_n||.$$

Now let $n \to \infty$.

(20) 10. (a) Suppose $f_n : [0,1] \to R^1$ is measurable for each $n = 1, 2, 3, \ldots$. Show that

$$E = \{x \in [0,1] : \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\}$$

is measurable.

Write

$$\begin{aligned} E =& \{x \in [0,1] : \{f_n(x), n \geq 1\} \text{ is Cauchy}\} \\ =& \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \left\{ x \in [0,1] : \Big| f_n(x) - f_m(x) \Big| \leq \frac{1}{k} \right\}. \end{aligned}$$

(b) Give an example of an uncountable family $f_\alpha : [0,1] \to R^1, \alpha \in [0,1]$, of measurable functions so that the function

$$f(x) = \sup_\alpha f_\alpha(x)$$

is not measurable.

Example: Let $P \subset [0,1]$ be non-measurable, and

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x = \alpha \in P; \\ 0 & \text{otherwise.} \end{cases}$$